$$\frac{\tilde{r}}{\bar{r}} \approx \ln(2) = 0.693$$

The last column of Table 5.2 shows the actual median to mean ratios for some selected values of *p*. This approximation is seen to be accurate to within a few percent for mean runlengths $\bar{r}$ of about 5.0 or larger.

Problem 5.13: An experienced 14.1 player knows that his mean runlength is 24.0 balls. What is the shot probability using the r_n estimate of the statistical distribution? What is the probability that this player will run between 50 and 75 balls? What is the probability of a run 100 or larger?

Answer: The individual shot probability for this player is p=24.0/25.0=.96. The probability of a run between 50 and 75, inclusive, is $r_{75}^{cum} - r_{49}^{cum}$=(1-$p^{76}$)-(1-$p^{50}$)= p^{50}-p^{76}=0.085. That is, the player should expect a run between 50 and 75 to occur in 8.5% of the attempts. The probability of a run of 100 or over is $1 - r_{99}^{cum}$=p^{100}=0.017; such a run will occur in 1.7% of the attempts.

Problem 5.14: This same 14.1 player is offered a friendly wager that for the rest of the day, every run over 20 balls he will win the wager amount, and every run of 20 balls or less he will lose the wager amount. Using the above statistical runlength model, is this a good proposition for the player?

Answer: Since his 24.0 mean is over 20 balls, it might seem at first that it would be a good proposition. However, upon closer inspection, the wager is really a matter of the median runlength, not the mean runlength. The individual shot probability for this player is p=24.0/25.0=.96, and this corresponds to a median runlength of $\tilde{r}$ =16 according to Table 5.2. The approximation from P5.12 gives $\tilde{r}$ ≈(0.693)(24.0)=16.6, so even if the player did not have the benefit of the table or a calculator to compute the exact median, he should expect to run 20 balls less than half of the time. Computation of the exact cumulative probability for m=20 gives $r_{20}^{cum} = 1 - 0.96^{21} = 0.576$, which means that he should expect to lose the wager 57.6% of the time, and win it only 42.4% of the time using the simple statistical model.

As shown in the following problem, safety play between opponents in an actual game situation skews the differences between the mean and the median runlengths even more than that predicted by this simple statistical model.

Problem 5.15: Assume that due to safety play by the opponent, the first shot of a player's inning has a success probability of only αp, with $0<\alpha\leq 1$, and each subsequent shot then has a success probability of p. What is the runlength probability distribution, the mean, the median, and the approximate ratio of the median to the mean as a function of α and p?

Answer: $r(\alpha)_0$=(1-αp), $r(\alpha)_1$=αpq, and, in general, $r(\alpha)_n$=$\alpha p^n q$ for n>0. Using the same approach as in P5.11, the mean runlength is found to be

$$\bar{r}(\alpha) = \frac{\alpha p}{1-p}$$

The cumulative probabilities are given by

$$r(\alpha)_m^{cum} = 1 - \alpha p^{m+1}$$

and the median runlength is determined by the smallest integer *m* that satisfies the equations

$$r(\alpha)_m^{cum} = 1 - \alpha p^{m+1} \geq \tfrac{1}{2}$$

$$m \geq -\left(\frac{\log(2\alpha)}{\log(p)} + 1\right)$$

The approximate ratio of the median to the mean, using the same approximations as in P5.12, is given by

$$\frac{\tilde{r}(\alpha)}{\bar{r}(\alpha)} \approx \frac{\ln(2\alpha)}{\alpha}$$

A few sample values for this ratio are shown in the following table.

α	1.0	0.9	0.8	0.7	0.6	0.5
$\tilde{r}(\alpha)/\bar{r}(\alpha)$	0.69	0.65	0.59	0.48	0.30	0.0

Only in the best possible case, α=1, is this ratio as good as that predicted in P5.12; in the other cases, this ratio becomes progressively worse with more aggressive safety play. (Note that in all of the equations above, setting α=1 produces agreement with the previous results.) This shows that even though the mean runlength is strongly dependent on safety play, the median runlength is even more sensitive.

The previous discussion concerned runlengths in which there were *n* successes followed by a single miss. In a game situation, this would apply to a single inning of a longer game. What is the runlength distribution after several innings? Using a similar approach as before, it is seen that the probability of accumulating *n* successful shots and *m* misses out of $N=m+n$ total shots is the binomial expansion term $P(p;n,m)$. The probability of a runlength score of exactly *n* after *m* innings (neglecting penalty points that might apply to the misses in the game) is given by

$$R_{nm} = qP(p;n,m-1) = \binom{n+m-1}{m-1} p^n q^m$$

That is, the first (m-1) misses can occur anywhere during the first ($n+m$-1) shots, but the last miss must occur on the last shot. It may be verified that $R_{n1}=r_n$ for all *n*, which is the single-inning runlength distribution that has been previously examined.

Problem 5.16: What is the mean score after *m* innings, using the R_{nm} distribution, as a function of *p*? What is the standard deviation of these scores?
Answer: The mean score is

$$\bar{R}_m = \sum_{n=0}^{\infty} nR_{nm} = \sum_{n=0}^{\infty} n\binom{n+m-1}{m-1}p^n q^m = q^m \sum_{n=1}^{\infty} \frac{(n+m-1)!}{(n-1)!(m-1)!} p^n$$

$$= mpq^m \sum_{n=0}^{\infty} \frac{(n+m)!}{(n)!(m)!} p^n = mpq^m \sum_{n=0}^{\infty} \binom{n+m}{m} p^n = \frac{mpq^m}{(1-p)^{m+1}}$$

$$= m\frac{p}{(1-p)} = m\bar{r}$$

The summation identity used in the above sequence may be verified using induction and repeated differentiation of the 1/(1-*p*) expansion as in P5.11. This result says simply that if a player has a mean, single-inning, runlength of $\bar{r}$, then after *m* innings, his mean score will be $m\bar{r}$.

The variance and standard deviation of the scores are

$$\sigma_m^2 = \left(\sum_{n=0}^{\infty} n^2 R_{nm}\right) - \bar{R}_m^2 = -\bar{R}_m^2 + \sum_{n=0}^{\infty} \left(n(n-1)+n\right)R_{nm}$$

$$= \bar{R}_m - \bar{R}_m^2 + \sum_{n=0}^{\infty} n(n-1)R_{nm} = \bar{R}_m - \bar{R}_m^2 + m(m+1)p^2 q^m \sum_{n=0}^{\infty} \frac{(n+m+1)!}{n!(m+1)!} p^n$$

$$= \bar{R}_m - \bar{R}_m^2 + \frac{m(m+1)p^2}{q^2} = \frac{mp}{q^2} = \frac{mp}{(1-p)^2}$$

$$\sigma_m = \sqrt{\sigma_m^2} = \frac{\sqrt{mp}}{q} = \frac{\sqrt{mp}}{1-p}$$

There are significant qualitative differences in the R_{nm} distributions (for a given inning count *m*) and the single-inning r_n distribution, particularly for large *m*. The most obvious of these is that the mode (or peak) of R_{nm} may occur for a nonzero value, as may be verified by inspection of a few examples; in contrast, the single-inning r_n distribution always peaks at *n*=0. The ratio of two successive probability values is

$$\frac{R_{n+1,m}}{R_{n,m}} = \left(\frac{n+m}{n+1}\right)p$$

The first factor is nonincreasing and approaches one from above as *n* increases, the second factor *p* is less than one, and the product may be either larger than one, indicating that the distribution is increasing towards a peak, or less than one, indicating that the distribution is decreasing after the peak. The mode is determined by the smallest nonnegative value of *n* for which the ratio is less than one, or zero if the ratio is always less than one. Solving the above ratio for this value gives the relation

$$Mode = Ceiling\left(\frac{mp-1}{1-p}\right) = Ceiling\left(\bar{R}_m - \frac{1}{q}\right)$$

The closed-form expression for the multiple-inning median involves special function evaluations (namely, the *incomplete beta function*), but it is easily determined by inspection of the numerical cumulative distributions for a given shot probability p and for a given value of the inning count m. The following table gives some examples of these statistical parameters for selected values of m and p.

Table 5.3. Multiple-Inning Runlength Statistics

	Mode					$\overline{R}_m$					$\tilde{R}_m$				
$p\backslash m$	1	2	4	8	16	1	2	4	8	16	1	2	4	8	16
.5	0	0	2	6	14	1.0	2.0	4.0	8.0	16.0	0	1	3	7	15
.6	0	1	4	10	22	1.5	3.0	6.0	12.0	24.0	1	2	5	11	23
.7	0	2	6	16	34	2.3	4.7	9.3	18.7	37.3	1	4	8	18	36
.8	0	3	11	27	59	4.0	8.0	16.0	32.0	64.0	3	7	15	31	63
.9	0	8	26	62	134	9.0	18.0	36.0	72.0	144.0	6	15	33	69	141

Because the median has no simple closed-form expression, as do the mode and the mean, a useful empirical approximation of the median is given by the weighted average

$$\tilde{R}_m = \tfrac{1}{3}\left(2\overline{R}_m + Mode\right) \approx \overline{R}_m - 1/(3q)$$

Comparison of this estimate with the above exact values in Table 5.3 shows that it is fairly accurate for m=4 or greater. Note that apart from the integer truncation in the mode and median evaluations, the differences between the mean, median, and the mode depend only on the individual shot probability p and are independent of the inning count m. This is why the three statistics appear to merge together in a relative sense in Table 5.3; they all increase with the inning count m but with constant differences.

When the multiple-inning runlength mean and the mode are relatively close to each other, indicating little skew, and the distribution has a single peak that is not close to zero, then the distribution is well approximated by a normal distribution with mean and standard deviation as determined in P5.16. That is, even though the single-inning distribution appears very different than a normal distribution, the multiple-inning distributions approach nonetheless a normal form as the inning count m increases. A perhaps simpler example of this common phenomenon is the point totals for two fair dice; each die individually has a flat distribution of point values from one to six, but when the totals are added for two dice, the probability of totaling to seven (p_7=1⁄6), which is the mean, is six times larger than rolling a two (p_2=1⁄36), with the other possible totals having intermediate probabilities. The results from P5.16 show that although the standard deviation of the score distributions are increasing with the inning count m, the deviation increases only as $\sqrt{m}$, whereas the mean score increases linearly as m. This means that the *relative deviations* (also called the *relative dispersions*), given by $\sigma_m / \overline{R}_m = 1/\sqrt{mp}$, decrease with respect to increasing inning count. When viewed in terms of percentages, the score will appear to more tightly cluster with larger inning counts, but when viewed in terms of actual score points, the score will appear to disperse more with larger inning counts; this is demonstrated in the following problem.

Problem 5.17: The 14.1 player from P5.13 plays 5 innings. What is his mean score, most likely score, and his median score? What is the standard deviation of the expected score distribution? What are these parameters after 50 innings?

Answer: This player's mean single-inning runlength is $\bar{r}$ =24.0, so his shot probability is p=0.96. After 5 innings, the mean score is given by

$$\bar{R}_5 = m\bar{r} = 5(24.0) = 120.0$$

His most likely score is the distribution peak, or the mode, given by

$$Mode = Ceiling\left(\bar{R}_m - 1/q\right) = Ceiling(120.0 - 25.0) = 95$$

The median is estimated as

$$\tilde{R}_5 = \tfrac{1}{3}\left(2\bar{R}_5 + Mode\right) \approx \bar{R}_5 - 1/(3q) = \bar{R}_5 - 25.0/3 = 112$$

This shows that the distribution of scores after 5 innings still is skewed significantly to the right. The standard deviation after 5 innings is

$$\sigma_5 = \sqrt{mp}/q = \sqrt{5(.96)}/.04 = 54.8$$

and the relative deviation is $\sigma_5/\bar{R}_5 = 54.8/120.0 = 0.457$, which is quite large.

If the player were to wager a fixed amount per point on the score after 5 innings, then the 120 point score demarks the fair betting point; those betting against the player at a lower score should expect to lose. But if someone were to wager in a betting pool of all possible scores, then a score of 95 is the most likely winner, and the scores close to 95 would be the best alternatives. And finally if the player were to wager simply whether the score is beyond a certain value after 5 innings (as in P5.14), then the 112 point score demarks the fair betting point; those betting against the player at a lower score should expect to lose.

For 50 innings the statistical parameters are: $\bar{R}_{50}$ =1200, *Mode*=1175, $\tilde{R}_{50}$ =1192, σ_{50}=173, and $\sigma_{50}/\bar{R}_{50}$ =0.144. This distribution is only slightly skewed to the right, and although the standard deviation is larger for 50 innings than for 5 innings, the relative deviation compared to the mean is much smaller.

In most of the previous discussion, the population distribution and statistical parameters have been assumed to be known, and the questions have been about the properties of various samples of this population. The reverse situation is now examined, namely how to predict the total population statistics from known sample statistics. For this purpose, consider a population from which all possible subsets of a particular size are formed. Each of these subsets has a mean, and the questions of interest are how reliable of an estimate for the total population mean is one of these subset means, how does this estimate depend on the standard deviation of the population, and how does this estimate improve with increasing sample size.

Assume that there is some population $\{x_i\}$ that has a mean $\bar{x}_p$ and a standard deviation σ_p. To simplify the following steps, it is assumed that the population is finite of size N, but the final results will also hold for infinite populations. Now suppose that m-element subsets are drawn with replacement from the population. There are N^m possible m-element subsets, and β is used as an index symbol to enumerate them. Such m-element sets are called *cartesian product* sets. The mean of each of these subsets is

$$\bar{x}_\beta = \frac{1}{m}\sum_{i=1}^{m} x_{\tau(i,\beta)}$$

where $\tau(i,\beta)$ is the population index of the *i-th* element of the β-*th* subset. The mean and variance of these m-element subset means is given by

$$\bar{x}_{[m]} = \frac{1}{N^m}\sum_{\beta=1}^{N^m} \bar{x}_\beta$$

$$\sigma[\bar{x}]_m^2 = \frac{1}{N^m}\sum_{\beta=1}^{N^m} \bar{x}_\beta^2 - \bar{x}_{[m]}^2$$

For m=1, there are N 1-element subsets, and the mean of each of these subsets is simply the value of that element, $\bar{x}_\beta = x_{\tau(i,\beta)} = x_i$, and the mean of the subset means is the same as the population mean.

$$\bar{x}_{[1]} = \frac{1}{N}\sum_{\beta=1}^{N} \bar{x}_\beta = \bar{x}_p$$

Similarly, the variance of these subset means is the same as the population variance.

$$\sigma[\bar{x}]_1^2 = \frac{1}{N}\sum_{\beta=1}^{N} \bar{x}_\beta^2 - \bar{x}_{[1]}^2 = \sigma_p^2$$

Now consider the situation in which the m-element subset parameters $\bar{x}_{[m]}$ and $\sigma[\bar{x}]_m^2$ are assumed to be available. The (m+1)-element subsets are constructed by forming the cartesian products $\{x_i \quad x_\beta ; i = 1 \quad N, \beta = 1 \quad N^m\}$. That is, each of the new set members is formed by combining the N population set elements with all possible N^m m-element subsets. The mean of each of these new subsets is

$$\bar{x}_{\beta'} = \bar{x}_{i\beta} = \frac{x_i + m\bar{x}_\beta}{m+1}$$

The mean of these subset means is

$$\bar{x}_{[m+1]} = \frac{1}{N^{m+1}}\sum_{\beta'=1}^{N^{m+1}} \bar{x}_{\beta'} = \frac{1}{(m+1)N^{m+1}}\sum_{\beta=1}^{N^m}\sum_{i=1}^{N}\left(x_i + m\bar{x}_\beta\right) = \frac{\bar{x}_p + m\bar{x}_{[m]}}{(m+1)}$$

This relation gives the results $\bar{x}_{[2]} = \bar{x}_p$, $\bar{x}_{[3]} = \bar{x}_p$, and, in general,

$$\bar{x}_{[m]} = \bar{x}_p$$

for all subset sizes *m*. The variance of the subset means is

$$\begin{aligned}
\sigma[\bar{x}]^2_{m+1} &= \frac{1}{N^{m+1}} \sum_{\beta=1}^{N^m} \sum_{i=1}^{N} \bar{x}^2_{i\beta} - \bar{x}^2_{[m+1]} \\
&= \frac{1}{N^{m+1}} \sum_{\beta=1}^{N^m} \sum_{i=1}^{N} \frac{1}{(m+1)^2} \left(x_i^2 + 2m x_i \bar{x}_\beta + m^2 x_\beta^2 \right) - \bar{x}_p^2 \\
&= \frac{1}{(m+1)^2} \left(\left(\frac{1}{N} \sum_{i=1}^{N} x_i^2 - \bar{x}_p^2 \right) + m^2 \left(\frac{1}{N^m} \sum_{\beta=1}^{N^m} \bar{x}_\beta^2 - \bar{x}_p^2 \right) \right) \\
&= \frac{\sigma_p^2 + m^2 \sigma[\bar{x}]^2_m}{(m+1)^2}
\end{aligned}$$

This relation gives the results $\sigma[\bar{x}]_2{}^2 = \sigma_p{}^2/2$, $\sigma[\bar{x}]_3{}^2 = \sigma_p{}^2/3$, and, in general, $\sigma[\bar{x}]_m{}^2 = \sigma_p{}^2/m$. The standard deviation of the subset means is then given by

$$\sigma[\bar{x}]_m = \frac{\sigma_p}{\sqrt{m}}$$

That is, the distribution of subset means becomes more narrowly peaked about the population mean as the size of the subset becomes larger. This relation also says that if the population standard deviation is small, then the mean estimates obtained from the subsets will be similarly sharp. In practice, the population standard deviation is usually not known, so it must be estimated from the *m*-element subspace statistics, along with the estimate of the mean.

When the subsets are formed by selection from the population without replacement, then the mean of the subset means is also given by

$$\bar{x}_{[m]} = \bar{x}_p$$

and the standard deviation of the subset means is given by

$$\sigma[\bar{x}]_m = \frac{\sigma_p}{\sqrt{m}} \sqrt{\frac{N-m}{N-1}}$$

The standard deviation of the means of the subsets formed without replacement are always smaller than those with replacement, and in particular $\sigma[\bar{x}]_m = 0$ when $m=N$. When $N >> m$, then the standard deviation of the mean is essentially the same for both types of subsets, and in the limit of an infinite population, both expressions are seen to be formally equivalent.

When the standard deviation is computed for an *m*-element sample space, then it is customary to use the factor (m-1) rather than *m* in the denominator; this has the effect of making the estimate for the population standard deviation slightly larger, but for reasonably large sample sizes the difference is unimportant. There are also other corrections that are sometimes applied when estimating population statistics from sample spaces. With the knowledge that such corrections can lead to slightly better estimations, they will not be used in the following examples for the sake of simplicity.

Problem 5.18: A population space has the values {1,2,4,5} which occur with equal probabilities. Compute the mean and standard deviation of the population set. Compute the mean and the standard deviation of the mean of the 2-element cartesian product set.
Answer: The population space mean is 3.0, and the population standard deviation is σ_p=sqrt(5/2)=1.5811. The 2-element cartesian product set is the same as the 2-element subsets drawn with replacement from the original set. There are 16 of these 2-element subsets, all with equal probability,

{1,1}	{1,2}	{1,4}	{1,5}
{2,1}	{2,2}	{2,4}	{2,5}
{4,1}	{4,2}	{4,4}	{4,5}
{5,1}	{5,2}	{5,4}	{5,5}

and with the corresponding means

1.0	1.5	2.5	3.0
1.5	2.0	3.0	3.5
2.5	3.0	4.0	4.5
3.0	3.5	4.5	5.0

The mean of these 2-element subset means is $\bar{x}_{[2]}$3.0, which demonstrates the general relation $\bar{x}_{[m]}=\bar{x}_p$. The standard deviation of these subspace means is $\sigma[\bar{x}]_2$=1.1180 which agrees with the equation $\sigma[\bar{x}]_2=\sigma_p$/sqrt(2). Note that these computations apply to a 4-element population space, or to a larger finite population space with the appropriate repetitions, or to an infinite population space with the appropriate probabilities.

Problem 5.19: A player has a practice routine that involves a particular sequence of shots. He keeps track of his numerical score for this routine for 10 weeks with the following results: {50, 44, 46, 52, 47, 51, 49, 45, 48, 50}. What is his mean score? Assuming a normal distribution of scores, what is the range of scores for which there is an 80% confidence level that the range includes the player's true mean score for this practice drill? The player experiments with a new technique (e.g. a different stroke technique) and scores a 53 on this drill, his highest score ever. Can the player be 95% certain that this is due to the technique change rather than to random chance? Can he be 90% certain that the score is due to the technique change? How does the sample size affect these assessments?
Answer: The mean score for the 10 weeks is 48.2. The standard deviation of the sample set is σ=2.52, which is taken as an approximation of the population standard deviation. The standard deviation of the mean is estimated as $\sigma[\bar{x}]_{10}=\sigma$/Sqrt(10)=0.797. Using the normal distribution approximation, Table 5.1 gives the critical value for an 80% confidence level as z_c=1.28. There is an 80% chance that the true mean score, which

would be the long-term mean of the player's score for this drill, is between $\bar{x}$ -$z_c\sigma[\bar{x}]$=47.2 and $\bar{x}$ +$z_c\sigma[\bar{x}]$=49.2.

The score of 53 is 4.8 points higher than the mean, or 4.8/2.52=1.90 standard units. The critical value for 95% confidence is z_c=1.96, so such a score would be expected to occur even with no change in stroke technique due to random chance within the 95% confidence predicted by a normal distribution assumption. For a 90% confidence z_c=1.645; such a score would not be expected to occur at this confidence level due simply to random chance. If the estimates for the mean and standard deviation were reliable (e.g. if the sample were much larger), then the player could say that he is 90% certain that the stroke technique change improved his score, but he could not say that he is 95% certain. However, this is a fairly small difference based on such a small sample. There is, after all, a good chance that the true mean is as high as 49.2, so a score of 53 is only 3.8 points, or 3.8/2.52=1.51 standard units, above the mean, and such a score can be expected to occur about 87% of the time due to random chance; that is, the player could say only that he is 13% confident that the score is due to the stroke change. Additional scores with the original technique would allow for a more accurate estimation of the population mean, and therefore a more accurate estimation of the effect of the stroke change in terms of standard units. This is one reason why players should establish practice routines and record their numerical scores over long periods of time; not only does it allow the player to track his progress, but it also allows for accurate statistical assessments of technique and equipment changes.

More exact determinations of the confidence can be achieved with additional data, including those obtained using the new stroke technique, by comparing the means and standard deviations of the different data sets (original stroke technique vs. the new one). When the means are sufficiently different, and when the standard deviations are sufficiently narrow, then there is a high confidence that the technique change is responsible for the score difference rather than simply the expected random fluctuations in the score. In general, the confidence is determined by the overlap regions of the two distribution tails. This type of comparative analysis can be quantified further with *chi-squared* tests (to compare expected and measured distribution statistics), the *Student's t-test* (to determine if two samples with the same variance have different means), the *F-Test* (to compare sample variances), and with *Analysis of Variance* techniques (to determine if two different samples actually are drawn from the same population). These methods are all outside the scope of this section, but they are mentioned in case the interested reader wishes to follow up on this interesting topic.

Statistical analysis can be used as a basis of choosing from among a set of possible tactics. This requires estimations of individual shot outcomes. A simple example of using probabilities to assess tactical options is a simple "one-ball" game, which occurs in actual game situations when, for example, both players are shooting at the 9-ball in a game of 9-ball, or when both players are shooting at the 8-ball in a game of 8-ball, or when both players are shooting at the black in a game of snooker, or when both players are shooting at the last ball in a game of one-pocket. In the one-ball game, a player is faced with a particular shot at a single ball. If he succeeds, then he wins immediately, and if he fails then the outcome depends on the outcome of the opponent's shot. As a way of keeping track of the details, such game situations may be represented with a diagram. The diagram corresponding to the one-ball game is shown in Fig. 5.2.

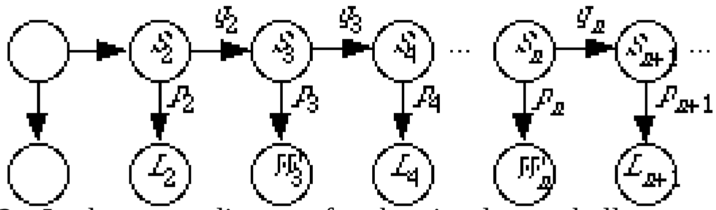


Fig. 5.2. In the game diagram for the simple one-ball game, there are an infinite number of nodes. Only the first few are shown explicitly. Player-1 wins at the terminal nodes *Wn* and he loses at the terminal nodes *Ln* . The transition probabilities are next to the connecting arcs.

In this diagram, the various *states* (or game situations) are called *nodes* and are shown by the circles. The possible state transitions are shown by the lines the connecting the nodes, and these lines are called *arcs*. The states labeled by *Sn* are where the winning ball has not been pocketed by the n^{th} inning, the states labeled by *Wn* are those where player-1 has won on the n^{th} inning, and the states labeled by *Ln* are those where player-1 has lost on the n^{th} inning. Each arc is associated with a particular transition probability. There are probability *weights* associated with each state. *S*1 is called the *graph head*, and P_{S1}=1 means that the *S*1 node is the starting point of the game. The weight of any other state is given by the sum of the weights of the previously connected state multiplied by the transition probability associated with the connecting arc. That is, the probability of arriving at a particular state in the game is the summation of the probability of arriving at all previous states, times the probability of making a transition from these previous states to the current state. The states labeled *Wn* and *Ln* are called *terminal nodes*, or probability *sinks*, or *tails*, since there are no arcs leaving these nodes; the game is over when the destination is one of these nodes. These game diagrams are a pictorial way of enumerating all possible paths as the probability density flows from the sources, through the transient states, to the probability sinks.

In the simple one-ball game depicted in Fig. 5.2, there is only one probability source *S*1, an infinite number of transient states *Sn*, and an infinite number of probability sinks *Wn* and *Ln*. There are only two arcs leaving each of the *Sn* nodes; in more complicated game situations, there may be several arcs leaving a node, each depicting a transition to a new possible state or to a previous state. The sum of all of the arc transition probabilities from a node is 1. In Fig. 5.2, the successful shots by either player are labeled p_n and the unsuccessful shots are labeled q_n with q_n=(1-p_n). This general idea of assigning probability weights to nodes, and to computing these weights from transition probabilities has already been used in the recursive algorithm for computing game score probabilities in P5.6.

In the general one-ball game, all of the individual p_n values will be different. It is interesting to consider some simpler cases in which the shot success probabilities are assumed to have special relations.

Problem 5.20: Assume that all of the shots taken by player-1 in the game depicted in Fig. 5.2 have a success probability of p_1, and all of the shots taken by player-2 have a success probability of p_2. In terms of these two parameters, what is player-1's total probability of winning, assuming that an infinite number of shots is allowed in the game? What combinations of p_1 and p_2 lead to a game probability of W=½?

Answer: The total chance of winning is the summation of the node weights P_{W1}, P_{W3},…. By multiplying the appropriate arc weights to get the node weights, the probabilities are given by $P_{W1}=p_1$, $P_{W3}=q_1q_2p_1$, $P_{W5}=(q_1q_2)^2p_1$, and so on. This will be called the two-parameter infinite-look-ahead approximation to the general one-ball game. The summation is

$$W^{[\infty]} = P_{W1} + P_{W3} + P_{W5} + \ldots = p_1 + q_1q_2p_1 + \left(q_1q_2\right)^2 p_1 + \ldots$$

$$= \sum_{i=1}^{\infty} P_{W(2i-1)} = \sum_{i=0}^{\infty} \left(q_1q_2\right)^i p_1 = \frac{p_1}{1-q_1q_2} = \frac{1}{1 + p_2\dfrac{\left(1-p_1\right)}{p_1}}$$

Setting $W^{[\infty]}$=½ and solving for p_2 in terms of p_1 gives

$$p_2^{crit} = \frac{p_1}{\left(1-p_1\right)}$$

When the actual value of p_2 is larger than this critical value, player-2 is expected to win, and when p_2 is smaller than this critical value then player-1 is expected to win.

A contour plot of $W^{[\infty]}$ as a function of the two parameters p_1 and p_2 is shown in Fig. 5.3. The region of the contour plot corresponding to small p_1 and large p_2 is the "sell-out" region; shots in this region should usually be avoided and other shots should be considered. The area of the contour plot corresponding to large p_1 and small p_2 is the "2-way shot" region; it is a great tactical advantage when these shots are available, as indicated by the large $W^{[\infty]}$ values. The area of the contour plot with small p_1 and small p_2 corresponds to defensive safety shots; the primary purpose is to keep the opponent from winning immediately, and to exploit any small advantage in probability over several innings. It is interesting to note how sensitive is the game probability estimate $W^{[\infty]}$ to small changes in the shot probabilities p_1 and p_2 in this region.

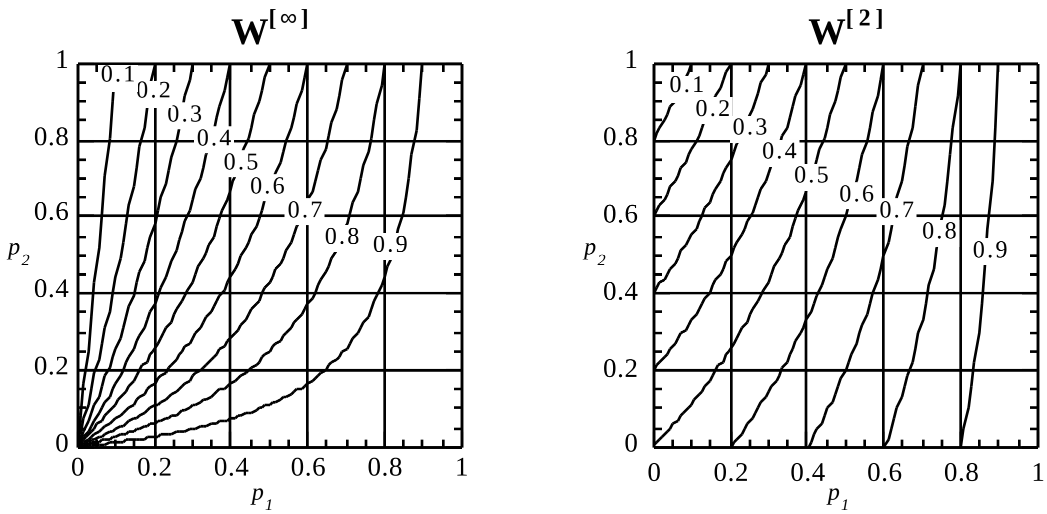


Fig. 5.3. Contour plots of the $W^{[\infty]}$ and $W^{[2]}$ approximations to the general one-ball game as a function of the two independent parameters p_1 and p_2. The W=½ contours are the same for both approximations.

A particularly important set of values corresponds to $W^{[\infty]}=½$. The p_1 and p_2 combinations for which $W^{[\infty]}>½$ are those in which player-1 is expected to win the simple one-ball game, and those values for which $W^{[\infty]}<½$ are those in which player-1 is expected to lose. In this simple game model, p_2 will be large either when player-2 is a very good shotmaker, or when the balls end up consistently in easy positions after a miss by player-1. It is clear from this graph that $W^{[\infty]}>½$ for all values of p_2 when $p_1>½$; this means that no matter how good of a shotmaker the opponent is, or how easy of a shot is left after each miss, player-1 is the expected winner when $p_1>½$. This is supported also by the p_2^{crit} expression given in P5.20. This advantage is afforded player-1 because he gets the first shot in the game. However, it is still useful to compare two possible strategies, even when both of them result in favorable outcomes for player-1. When $p_1<½$, then the expected outcome clearly depends on p_2; when p_2 is sufficiently small, then player-1 is still the expected winner, but when p_2 is large, then player-1 is expected to lose.

Problem 5.21: Assume that player-1 in the one-ball game takes his first shot with a success probability of p_1, and that a good estimate of the value of p_2 is known, but after these first two shots both players are assumed to have an even chance of winning the game. In terms of these two parameters, what is player-1's total probability of winning? What combinations of p_1 and p_2 lead to a game probability of W=½?

Answer: The game diagram for this approximation to the general one-ball game is shown in Fig. 5.4. This will be called the two-parameter two-shot-look-ahead approximation. There are now only two nodes in Fig. 5.4 that correspond to wins for player-1. The game probability is the sum of the weights for these two nodes.

$$W^{[2]} = P_{W1} + P_W = p_1 + \tfrac{1}{2} q_1 q_2$$

A contour plot of $W^{[2]}$ is shown in Fig. 5.3. When compared to the $W^{[\infty]}$ contour plot, it is seen that the two approximations give similar, but not exactly equivalent, predictions of game probabilities.

Rearranging the $W^{[2]}=½$ equation to solve for p_2 as a function of p_1 gives

$$p_2^{crit} = \frac{p_1}{(1 - p_1)}$$

which is the same curve of critical values as determined previously for $W^{[\infty]}$.

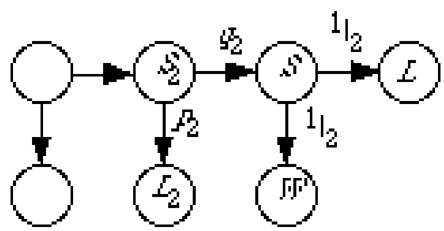


Fig. 5.4. The two-shot-look-ahead approximation to the general one-ball game depends on only two independent parameters p_1 and p_2 that characterize the shot success for the first two innings. After the second inning, the game outcome probability is split equally between the two players.

If three independent shot parameters are known, then this leads to a three-shot-look-ahead approximation, and in general there are *n*-shot-look-ahead approximations involving *n* independent probability parameters. It should be stressed that both the two-parameter infinite-look-ahead and the two-parameter two-shot-look-ahead equations are approximations to the general one-ball game; one of these should not be regarded as an approximation to the other. In some situations, the infinite-look-ahead assumption may be more appropriate, while in other situations the two-shot-look-ahead assumption might be best. For example, if player-1 is a weaker player than player-2, then from a relatively neutral position the infinite-look-ahead model with a large p_2 value would provide the most reliable estimates of game outcomes; but if both players are roughly equal in ability, and if player-1 has a decided positional advantage for his first shot (e.g. a strong 2-way shot corresponding to a large p_1 and a small p_2), then the two-shot-look-ahead model would provide the most reliable estimates of game outcomes.

If two players are playing a multigame 9-ball match (or, for example, 8-ball or one-pocket), then the opening break shot is usually regarded as an advantage. Matches are sometimes played in which the winner of each game is rewarded by being allowed the opening break in the next game (*winner-breaks*), or they may be played where the loser of one game breaks in the next game (*loser-breaks*), or the players may *alternate breaks* from game to game, or one of the players may break all of the games (e.g. *player-1* *breaks*; this is usually regarded as a handicap advantage for the breaking player to compensate for some difference in skills). This last situation is interesting in the context of the above analysis of the one-ball game. Suppose that player-1 breaks each game, and that he wins each of these games on his first inning with a probability of p_1. In the second inning, player-2 wins with a probability of p_2, and so on. The game diagram for this situation is the same as for the one-ball game. The nodes of the diagram correspond to inning counts rather than individual shots, and the winning probabilities are with respect to games rather than individual shot successes, but the mathematical structure is the same for both situations. The infinite-look-ahead and the two-shot-look-ahead approximations, and the discussions of these two parameters in P5.20 and P5.21 apply also, in a perhaps more approximate way, to this multigame match situation. With these approximations, the contour plots in Fig. 5.3 show that player-1 would be expected to have an advantage over player-2 by virtue of playing the first inning, and this advantage becomes more significant for larger values p_1.

Problem 5.7: Using the game probability estimates $W^{[\infty]}$ and $W^{[2]}$, compute the probability for player-1 to win when (p_1,p_2) have the values: (0.1,0.9), (0.9,0.1), (0.9,0.9), (0.5,0.5), (0.25,0.33), (0.4,0.8), and (0.3,0.3).

Answer: The game probability estimates are given in the following table

p_1	p_2	$W^{[\infty]}$	$W^{[2]}$
0.1	0.9	0.11	0.15
0.9	0.1	0.99	0.95
0.9	0.9	0.90	0.90
0.5	0.5	0.67	0.62
0.25	0.33	0.50	0.50
0.4	0.8	0.45	0.46
0.3	0.3	0.59	0.55

The (p_1,p_2)=(0.1,0.9) shot is a sell-out shot. Player-1 is expected to lose this game, even with the first-shot advantage; he should consider another choice of shot. The (0.9,0.1) situation is a strong 2-way shot; player-1 is the favorite in this game. For (0.9,0.9), player-1 is again the favorite. Even though p_1 for this case is the same as the previous one, it is clear that it is better to plan to leave a low-percentage shot for the opponent than a high-percentage one (i.e., $W^{[\infty]}$=0.99 is better than $W^{[\infty]}$=0.90). For the (0.5,0.5) shot, player-1 is the expected game winner using both estimates, even though he and his opponent are evenly matched with equally difficult shots; this is due to the first-shot advantage. The (0.25,0.33) shot corresponds to a (p_1,p_2^{crit}) pair, so each player has equal probability of winning according to both estimates.

Player-1 has a disadvantage at (0.4,0.8) and a fairly significant advantage at (0.3,0.3) using both estimates of the game probability. It is interesting to compare these last two situations, since it appears to be a paradox to many inexperienced pool players. In both cases, the individual shot probability p_1 is relatively small. In fact, p_1 is smaller for the second (favorable) game outcome than for the first (unfavorable) game outcome. In the first case, he leaves a high-probability shot for his opponent, while in the second case he leaves a low-probability shot. Inexperienced players often choose shots based only on their estimate of the first-shot success probability, that is only on p_1. This is an example of how the down-side consequences (what occurs after the miss) outweigh the up-side reward (which shot has the higher p_1). In other words, it is sometimes more important not to "sell out" than it is to try to succeed with a spectacular shot. In "tactic-rich" games involving relatively difficult shots, such as one-pocket, this kind of decision is part of the routine shot-selection process. The simple one-ball game mathematical model used here provides an approximate way to quantify the relative importance of the up-side reward and the down-side consequences for these more complex situations in actual pool games.

In physical simulations, processes that may be characterized by probabilities are called *stochastic* systems, and an important class of stochastic systems, called *Markov* processes, are those in which the probability of making a transition from one state to another depends only on the initial and final states, and not upon a history of the previous states. The game diagrams described above are examples of Markov processes. One way to analyze these types of diagrams is to consider them as a "time dependent" process. In some situations, it is the transient short-time behavior that is of importance, and at other times it is the long-time steady-state behavior that is most interesting. In the above game diagrams, the "time" parameter corresponds to the inning count, or to the shot count, or, as will be discussed below, to a game count. An initial probability distribution is assigned to the nodes of the graph, and this probability density flows through the graph as individual time steps are taken. The information that is most important in the pool-game situation is how much of this probability density ends up in the various terminal nodes. In the above examples of game diagrams, it was possible to answer this question by recognizing relatively simple algebraic simplifications that allowed closed-form expressions to be obtained. But in more complicated situations, such closed-form expressions may not be apparent, or they may not even exist. In these situations, it is still possible to extract the long-time steady state probability densities numerically, and this general procedure is now discussed.

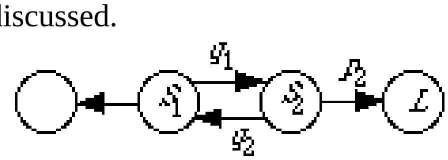


Fig. 5.5. The game diagram for the two-parameter infinite look-ahead approximation to the general one-ball game consists of four nodes: two terminal nodes and two transition nodes.

This procedure will be applied first to the one-ball game so that the results can be compared to those previously found. The two-parameter infinite look-ahead game probability will be examined. For this purpose, it is convenient to use a simpler game diagram, shown in Fig. 5.5, that has only a finite number of nodes (four in this case). In this diagram, all of the wins for player-1 are treated equivalently, with a single terminal node, and likewise all loses for player-1 are treated with a single terminal node. The two transient states are related simply to which player is shooting the shot. The important quantity in Markov analysis is the *probability transition matrix*. The rows and columns of this matrix correspond to the states of the system, and therefore to the nodes of the game diagram. The element M_{ij} corresponds to the arc weight of the arc that connects node j to node i; that is M_{ij} is the probability of making a transition from the state corresponding to node j to the state corresponding to node i. Nodes that are not connected correspond to zero M_{ij} values. Terminal nodes are assumed to make transitions to themselves each time step with unit probability. The transition matrix corresponding to the two-parameter infinite-look-ahead approximation to the one-ball game is

$$\mathbf{M} = \begin{pmatrix} 1 & p_1 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ 0 & q_1 & 0 & 0 \\ 0 & 0 & p_2 & 1 \end{pmatrix}$$

The rows and columns of the matrix correspond to the nodes *W*, *S1*, *S2*, and *L*, respectively. The sum of the elements in a column of **M** is one, which reflects the fact that probability density is not destroyed by taking a time step. The vector-matrix product relation (1,1,1,1)**M**=(1,1,1,1) is a consequence of this, and such a relation is always satisfied for a Markov transition matrix. This means that there exists at least one left eigenvector of **M** that corresponds to an eigenvalue of one, and therefore there also must exist a right eigenvector with this same eigenvalue; the existence of this unit eigenvalue is important in this analysis. Let κ=(1,1,1,1) be this left eigenvector and $\mathbf{v}^0$ be an arbitrary column vector, then the dot product $\kappa\mathbf{v}^0$ is equal to the sum of the elements of $\mathbf{v}^0$. The vectors of interest correspond to probability densities, and such vectors contain only nonnegative elements that sum to one. A single time-propagation step from an initial vector $\mathbf{v}^0$ is given by the matrix-vector product $\mathbf{v}^1=\mathbf{M}\mathbf{v}^0$. Operating on the left of this equation with κ gives the result $\kappa\mathbf{v}^1=\kappa\mathbf{v}^0=1$, which means that the sum of the probability density after the time step is the same as the sum before the time step. After two steps, the density is given by $\mathbf{v}^2=\mathbf{M}\mathbf{v}^1=\mathbf{M}^2\mathbf{v}^0$, and after n steps the density is given by $\mathbf{v}^n=\mathbf{M}\mathbf{v}^{(n-1)}=\mathbf{M}^n\mathbf{v}^0$. Operating on the left by κ on any of these relations shows that the total density is conserved always by the propagation operations.

What does the vector $\mathbf{v}^n$ look like after a large number of steps? The answer depends on the eigenvalues of the matrix **M**. In general the right eigenvectors of **M** satisfy the equation $\mathbf{MR}=\mathbf{R}\lambda$ in which the right eigenvectors form the columns of **R**, and the diagonal matrix λ contains the corresponding eigenvalues. This allows the matrix **M** to be written as $\mathbf{M}=\mathbf{R}\lambda\mathbf{R}^{-1}$. The matrix $\mathbf{M}^2$ is given by $\mathbf{M}^2=\mathbf{R}\lambda\mathbf{R}^{-1}\mathbf{R}\lambda\mathbf{R}^{-1}=\mathbf{R}\lambda^2\mathbf{R}^{-1}$, and in general $\mathbf{M}^n=\mathbf{R}\lambda^n\mathbf{R}^{-1}$ with

$$\lambda^n = \begin{pmatrix} \lambda_1^n & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 \\ 0 & 0 & \lambda_3^n & 0 \\ 0 & 0 & 0 & \lambda_4^n \end{pmatrix}$$

This expression allows the probability distribution after an arbitrary number of time steps to be determined with relatively little effort, compared to the straightforward approach using repeated multiplications. The rows of the matrix $\mathbf{L}=\mathbf{R}^{-1}$ are the left eigenvectors, and, with the appropriate choice of normalization, one of these rows is $\kappa=\mathbf{L}^{[1]}$. If $\lambda_i=1$, then $\lambda_i^n=1$ for all n, and if $|\lambda_i|<1$, then $\lambda_i^n \to 0$ as $n \to \infty$. For the transition matrices associated with game diagrams, the eigenvalues are $-1<\lambda_i\leq 1$. This allows the vector limit after an infinite number of time steps, called the *stochastic limit*, to be written as

$$\mathbf{v}^\infty = \mathbf{M}^\infty\mathbf{v}^0 = \mathbf{R}\lambda^\infty\mathbf{R}^{-1}\mathbf{v}^0 = \mathbf{R}\lambda^\infty\mathbf{L}\mathbf{v}^0 = \sum_i^{(\lambda_i=1)} \mathbf{R}_i\left(\mathbf{L}_i\mathbf{v}^0\right)$$

in which the summation includes only the right and left eigenvector pairs that correspond to eigenvalues of unit magnitude. In many cases, there is only a single eigenvalue in this summation, and in this case $\mathbf{v}^\infty=\mathbf{R}_1(\kappa\mathbf{v}^0)=\mathbf{R}_1$, where $\mathbf{R}_1$ is the right eigenvector associated with the single eigenvalue of unit magnitude (and scaled appropriately to conserve density). In this case there is a single stochastic limit $\mathbf{v}^\infty$ that is approached for any arbitrary starting density $\mathbf{v}^0$. In other cases, there may be more than one such vector in the summation, in which case the final stable probability distribution depends on the starting distribution $\mathbf{v}^0$. It may be verified that $\mathbf{M}^\infty\mathbf{M}^\infty=\mathbf{M}^\infty$, and therefore $\mathbf{M}^\infty$ is a projection operator; it operates upon an arbitrary probability density distribution and projects this vector onto the subspace of the stable state distribution(s).

In the specific case of the two-parameter infinite-look-ahead approximation to the one-ball game, the eigenvalues of $\mathbf{M}$ are $(1,1,\sqrt{q_1q_2},-\sqrt{q_1q_2})$. There are two eigenvalues of unit magnitude. The matrix $\mathbf{M}^\infty$ is given by

$$\mathbf{M}^\infty = \begin{pmatrix} 1 & \frac{p_1}{1-q_1q_2} & \frac{p_1q_2}{1-q_1q_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{q_1p_2}{1-q_1q_2} & \frac{p_2}{1-q_1q_2} & 1 \end{pmatrix}$$

Problem 5.23: Given an initial probability distribution of $\mathbf{v}^0=(0,1,0,0)^T$, compute $\mathbf{v}^1$, $\mathbf{v}^2$, $\mathbf{v}^3$, $\mathbf{v}^4$, $\mathbf{v}^5$, and $\mathbf{v}^\infty$. What are these same vectors for $\mathbf{w}^0=(0,0,1,0)^T$? What is the stochastic limit for the vector $(0,½,½,0)^T$? What is the meaning of these three limits?
Answer: For $\mathbf{v}^0=(0,1,0,0)^T$, the first few vectors are

$$\mathbf{v}^0=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \mathbf{v}^1=\begin{pmatrix}p_1\\0\\q_1\\0\end{pmatrix},\ \mathbf{v}^2=\begin{pmatrix}p_1\\q_1q_2\\0\\q_1p_2\end{pmatrix},\ \mathbf{v}^3=\begin{pmatrix}p_1(1+q_1q_2)\\0\\q_1^2q_2\\q_1p_2\end{pmatrix},\ \mathbf{v}^4=\begin{pmatrix}p_1(1+q_1q_2)\\(q_1q_2)^2\\0\\q_1p_2(1+q_1q_2)\end{pmatrix},$$

$$\mathbf{v}^5=\begin{pmatrix}p_1\left(1+q_1q_2+(q_1q_2)^2\right)\\0\\q_1^3q_2^2\\q_1p_2(1+q_1q_2)\end{pmatrix},\text{ and }\mathbf{v}^\infty=\mathbf{M}^\infty\mathbf{v}^0=\begin{pmatrix}\frac{p_1}{1-q_1q_2}\\0\\0\\\frac{q_1p_2}{1-q_1q_2}\end{pmatrix}.$$

The first element of these vectors, which corresponds to the probability that player-1 will win the game after the appropriate number of innings, is seen to be the same as the series of cumulative probabilities computed in P5.20, and the corresponding element of the $\mathbf{v}^\infty$ vector agrees also with that from P5.20. Although the 4-node game diagram in Fig. 5.5 seems simpler than the infinite-node diagram in Fig. 5.2, the step-by-step propagation of the 4-node density vector gives the same information as the more complicated game diagram. It may also be noted in this example that the sum of the densities for the four nodes always adds up to 1.

For $\mathbf{w}^0=(0,0,1,0)^T$, the first few vectors are

$$\mathbf{w}^0=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \mathbf{w}^1=\begin{pmatrix}0\\q_2\\0\\p_2\end{pmatrix},\ \mathbf{w}^2=\begin{pmatrix}p_1q_2\\0\\q_1q_2\\p_2\end{pmatrix},\ \mathbf{w}^3=\begin{pmatrix}p_1q_2\\q_1q_2^2\\0\\p_2(1+q_1q_2)\end{pmatrix},\ \mathbf{w}^4=\begin{pmatrix}p_1q_2(1+q_1q_2)\\0\\(q_1q_2)^2\\p_2(1+q_1q_2)\end{pmatrix},$$

$$\mathbf{w}^5=\begin{pmatrix}p_1q_2(1+q_1q_2)\\q_1^2q_2^3\\0\\p_2\left(1+q_1q_2+(q_1q_2)^2\right)\end{pmatrix},\text{ and }\mathbf{w}^\infty=\mathbf{M}^\infty\mathbf{w}^0=\begin{pmatrix}\frac{p_1q_2}{1-q_1q_2}\\0\\0\\\frac{p_2}{1-q_1q_2}\end{pmatrix}.$$

For the $(0,\frac{1}{2},\frac{1}{2},0)^T$ vector, the stochastic limit is

$$\mathbf{M}^\infty\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}=\tfrac{1}{2}\mathbf{M}^\infty\left(\mathbf{v}^0+\mathbf{w}^0\right)=\tfrac{1}{2}\begin{pmatrix}\frac{p_1(1+q_2)}{1-q_1q_2}\\0\\0\\\frac{p_2(1+q_1)}{1-q_1q_2}\end{pmatrix}$$

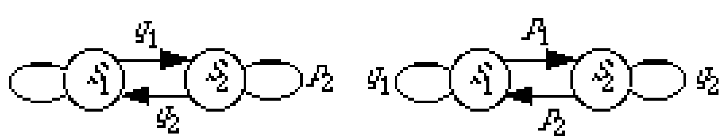
For the $\mathbf{v}^0$ case, player-1 is given the first shot of the game, and consequently the first chance to win, whereas in the $\mathbf{w}^0$ case player-2 is given the first shot. For this game diagram, there are two eigenvalues equal to one, so there are two possible independent, asymptotic stable solutions for these two initial conditions, $\mathbf{v}^\infty$ and $\mathbf{w}^\infty$. The same game diagram, and the same Markov analysis, covers both situations. The $(0,\frac{1}{2},\frac{1}{2},0)^T$ initial probability distribution corresponds to the alternating break situation, or perhaps to some other situation in which each player has the first shot an equal number of times.

In this simple approximation to the one-ball game, closed-form expressions could be found for all of the eigenvalues and eigenvectors, but in more complicated game situations this may not be true. If only a numerical solution is possible, then the asymptotic stable solutions can be found for any initial density vector by computing the eigenvalues and eigenvectors of the transition matrix numerically.

In the winner-breaks match situation, a player with a strong break advantage has the opportunity to break and win several consecutive games in a row. This may occur because the player breaks and runs, never allowing the opponent to have a shot, or it may be because the player is a good tactical player and he never allows his opponent an open shot on a winnable table. The winner-breaks match situation amplifies the break advantage in this case through a positive feedback situation. In a loser-breaks match, the player with the break advantage cannot exploit it because when he wins one game, his opponent gets to break, nullifying the break advantage. In this case the break advantage is damped through a negative feedback situation. The following problem shows how this feedback situation can be quantified.

Problem 5.24: Two players are playing a series of 9-ball games, and the winner of one game breaks in the subsequent game. When player-1 breaks, player-1 wins with a probability of p_1. When player-2 breaks, player-2 wins with a probability of p_2. What fraction of the total games will player-1 win if a large number of games are played? If the loser of one game breaks in the next game, what fraction of the total games will player-1 win? What is the expected outcome in the alternating break situation?

Answer: The game diagrams for the winner-breaks (WB) and for the loser-breaks (LB) situations are:



There are two states of interest, *S*1 is when player-1 breaks and *S*2 is when player-2 breaks. The corresponding transition matrices are:

$$\mathbf{M}^{WB} = \begin{pmatrix} p_1 & q_2 \\ q_1 & p_2 \end{pmatrix}, \quad \mathbf{M}^{LB} = \begin{pmatrix} q_1 & p_2 \\ p_1 & q_2 \end{pmatrix}$$

For both of these situations, closed-form solutions can be found for the eigenvalues and eigenvectors. The eigenvalues for the two cases are ($1, p_1+p_2-1$), and ($1, 1-p_1-p_2$), respectively. In both cases, there is only a single eigenvalue of unit magnitude, so each situation has a single asymptotic distribution given by $\mathbf{v}^{\infty}=\mathbf{R}_1$ for any choice of initial density $\mathbf{v}^0$. This means that it does not matter which player has the initial break in the match; the long-run winner is determined only by the two probability parameters p_1 and

p_2 and by the match type, WB, LB, or AB. For the WB and LB cases, the $\mathbf{R}_1$ eigenvectors are, respectively,

$$\mathbf{R}_1^{WB} = \begin{pmatrix} \frac{q_2}{q_1+q_2} \\ \frac{q_1}{q_1+q_2} \end{pmatrix}, \text{ and } \mathbf{R}_1^{LB} = \begin{pmatrix} \frac{p_2}{p_1+p_2} \\ \frac{p_1}{p_1+p_2} \end{pmatrix}$$

In the WB case, player-1 wins a fraction of games corresponding to $W^{WB}=R_{11}{}^{WB}=q_2/(q_1+q_2)$; in the LB case, the fraction of games won by player-1 is $W^{LB}=R_{21}{}^{LB}=p_1/(p_1+p_2)$.

In the alternating-break situation, player-1 breaks half of the games, and proceeds to win the fraction p_1 of these, and player-2 breaks the other half of the games, and player-1 wins the fraction q_2 of these games. The total fraction of games won by player-1 is therefore $(p_1+q_2)/2$ in the alternating break situation.

Contour plots of the winning probabilities W^{WB}, W^{LB} and W^{AB} determined from P5.24 are shown as a function of p_1 and p_2 in Fig. 5.6. It is surprising how different these plots appear. It may be verified that the critical values of p_1 and p_2 that correspond to $W=½$ are the same in the WB, LB, and AB matches, namely $W=½$ when $p_1=p_2$ in all three cases; this means that the break choice is not expected to change the eventual winner, provided a large number of games are played in the match; however the margin by which the winner is expected to win can depend in a significant way on the match format due to the interplay between the positive and negative feedback effects. This means that in a match that is handicapped by payout stakes can depend in a significant way on the break choice. When $p_1=q_2$, then the player-1 game probability does not depend on which player breaks, and there is no break advantage or disadvantage; this relation is equivalent to $p_1+p_2=1$, and it may be verified that $W^{WB}=W^{LB}=W^{AB}=p_1$ in all three match situations when this condition is satisfied.

A contour plot of the difference probability, $W^{diff}=W^{WB}-W^{LB}$, is also shown in Fig. 5.6. The solid positive contour lines correspond to the situations in which player-1 has the best chance in a WB match, and the dashed negative contour lines correspond to the situations in which player-1 has the best chance in a LB match. It is clear in this figure that there can be significant differences in the outcomes of the WB and LB matches. In most situations, the individual probabilities are expected to be close to 0.5 for both players, and the difference contour plot in Fig. 5.6 shows that the break choice makes only a small difference in the outcome in these situations. However, the most drastic differences occur when one of the players has a strong break advantage or disadvantage. Table 5.4 gives the player-1 winning probabilities for a few selected values of p_1 and p_2 that demonstrate these trends.

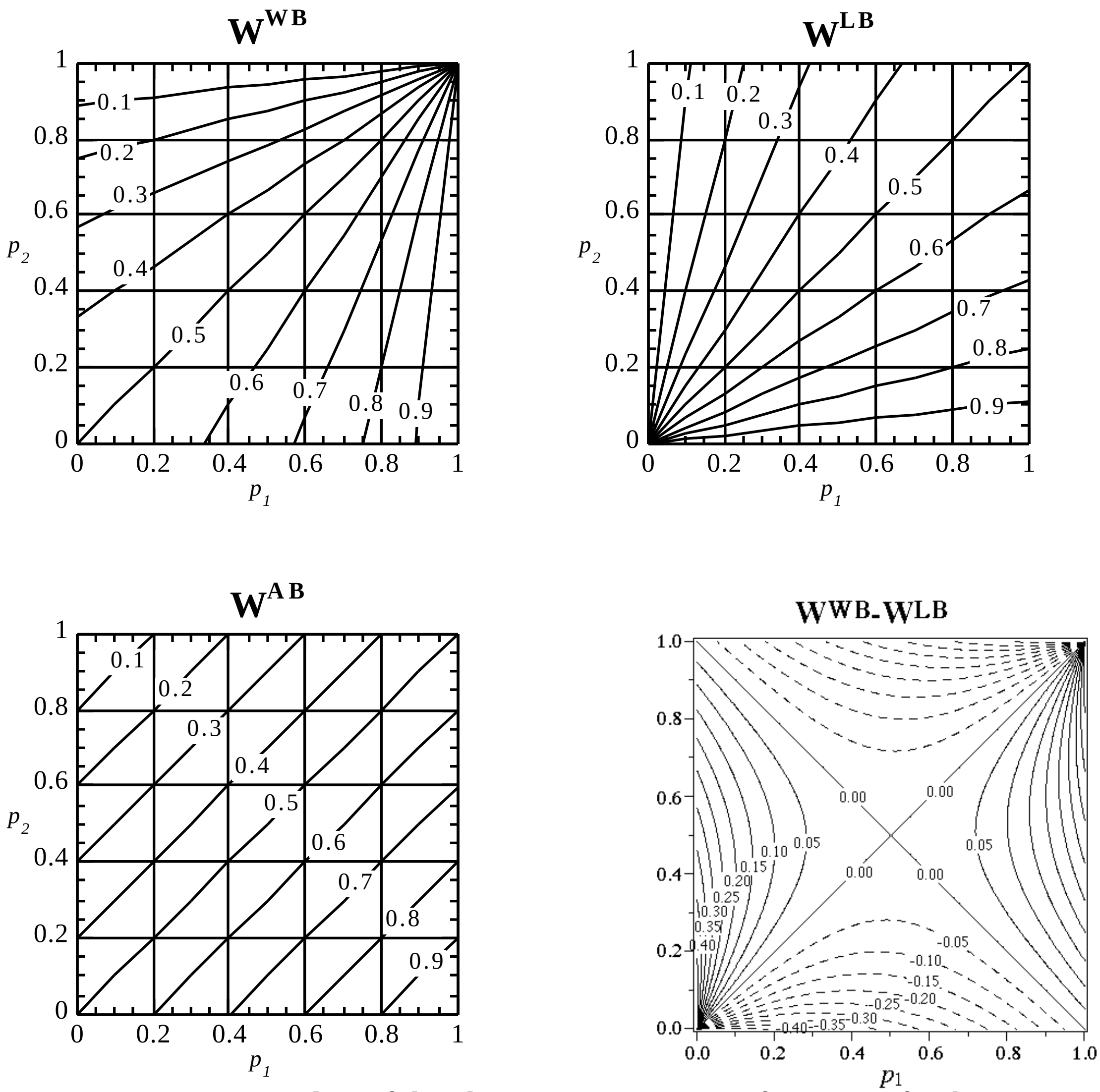


Fig. 5.6. Contour plots of the player-1 winning game fraction W for three types of matches: winner-breaks, loser-breaks, and alternating breaks. The p_1 parameter is the fraction of games that player-1 wins when player-1 breaks, and p_2 is the fraction of games that player-2 wins when player-2 breaks. When the contour lines are closely spaced, then the winning fraction W is very sensitive to small changes in the parameters p_1 and p_2.

The first three rows of Table 5.4 correspond to equally matched players. In the first row, both players might be weak runout players, or weak tactical players, who tend to lose most of the games that they break, and both players have a break disadvantage; in the second row both players win the same percentage of games that they lose when they break, and neither player has a break advantage or a break disadvantage; in the third row, both players might be strong runout players, or strong tactical players who tend to control the table once they get a shot, and the break is an advantage that can be exploited by both players. Matches between equal-strength players are expected to be even in WB, LB, and

AB situations, as seen in the first three rows. The next two rows show the expected result from a strong mismatch; in the fourth row player-1 is the underdog, and in the fifth row he is the strong favorite. Since $p_1+p_2=1$ in these two cases, there is no break advantage and the winning expectations are equal for all three match situations.

Table 5.4. Comparison of WB, LB, and AB winning probabilities.

Row	p_1	p_2	W^{WB}	W^{LB}	W^{AB}
1	.1	.1	.5	.5	.5
2	.5	.5	.5	.5	.5
3	.9	.9	.5	.5	.5
4	.1	.9	.1	.1	.1
5	.9	.1	.9	.9	.9
6	.5	.1	.64	<u>.83</u>	.7
7	.5	.9	.17	<u>.36</u>	.3
8	.9	.5	<u>.83</u>	.64	.7
9	.1	.5	<u>.36</u>	.17	.3
10	.95	.94	<u>.54</u>	.50	.505

Rows six and seven are two cases in which moderate mismatches occur and in which player-1 is the medium-strength player. In row six, he is favored to win over a weak opponent in both WB and LB match situations, but he is expected to win a much higher fraction of games in the LB match than in the WB match. This is because in the LB situation, player-1 can win one game and then take advantage of his opponent's breakshot weakness immediately in the next game by forcing him to break. The AB win fraction is between those of the WB and LB, and this trend holds for all combinations of the parameters p_1 and p_2. In row seven, player-1 is a medium strength player playing against a strong player; he is expected to lose in all three types of match situations, but his game percentage is about twice as large in the LB match as in the WB match. This is because he breaks more often than his stronger opponent in the LB situation, and although he does not benefit particularly from his own breaks, he keeps his opponent from exploiting his break advantage. The last two rows show the same types of mismatches as rows six and seven, but with the assumption that player-1 is the strong player (row eight) or the weak player (row nine) against a medium strength player; in both cases, a WB match situation is most beneficial to player-1. In row eight, player-1 benefits in the WB situation by exploiting his break advantage. In row nine, player-1 is actually penalized by being forced to break, and he breaks fewer times in the WB situation than in the LB situation which helps him limit his loses.

Row ten shows the expected results for two strong players who are closely, but not exactly, matched. Player-1 has a very slight 1% win-while-breaking game probability advantage over that of his opponent. It is interesting that in the WB match, this small advantage is magnified into a 4% difference in the expected game fraction, whereas in the LB match, the effect of this small advantage is almost eliminated in the expected game fraction. In the AB match, the 1% p advantages gets diluted to a 0.5% W advantage. The amplification of small differences of WB matches is a consequence of the clustering of the contour lines in the upper right corner of the W^{WB} graph in Fig. 5.6. Similarly, the damping out of such differences in the LB situation is a consequence of the wide spacing of the contour lines in the upper right corner of the W^{LB} graph. When player-1 has a slight breakshot advantage over his strong opponent, then he should prefer the WB situation, but when he has a slight breakshot disadvantage compared to his strong opponent, then he should prefer the LB situation.

Problem 5.25. Two players play a stakes-handicapped match in which player-1 wins 1.0 points for each game that he wins and he loses 2.0 points for each game that he loses. The win-while-break percentages for the two opponents are p_1=0.9 and p_2=0.5. What is the expected outcome for WB, LB, and AB matches?

Answer: The expectation of return R by player-1 for each game is given by

$$R = W\ Z_W - L\ Z_L$$

where W is the probability of winning each game, Z_W is the number of points that he wins for each of these games, L=(1-W) is the losing game probability and Z_L is the number of points that he loses. Using the results from Table 5.4, it is seen that R^{WB}=(.83)(1.0)-(.17)(2.0)=0.49, R^{LB}=(.64)(1.0)-(.36)(2.0)=-0.08, and R^{AB}=(.70)(1.0)-(.30)(2.0)=0.10. Player-1 is expected to win in the WB match and AB match situation, but he is expected to lose in the LB match situation.

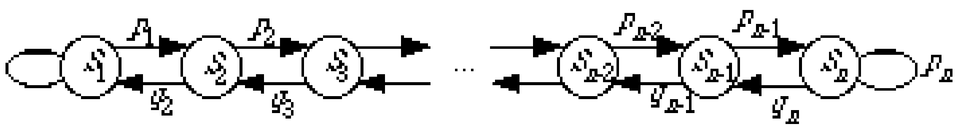


Fig. 5.7. The general game diagram for a progressive drill has n nodes and n independent probability parameters.

A progressive practice drill has a difficulty parameter that can be increased or decreased. A difficulty parameter might be a shot angle, or a shot distance, or some cue ball position goal, the number of object balls, or some combination of such parameters. In a progressive practice drill, when the player succeeds at one level of difficulty, then he is rewarded by being allowed to attempt the next level of difficulty; when the player fails at a level of difficulty, then he is penalized by being forced back to the previous level. Suppose that there are n levels of difficulty, numbered 1…n. Failure at the first level means that the player attempts that level again, and success at the n^{th} level means that level-n is attempted again. Suppose that the probability of success at the i^{th} level is denoted p_i, and the failure probability is therefore q_i=1-p_i. The game diagram for a general progressive practice drill is shown in Fig. 5.7. The Markov transition matrix for such a progressive drill has the form

$$\mathbf{M} = \begin{pmatrix} q_1 & q_2 & 0 & \cdots & 0 \\ p_1 & 0 & \ddots & \cdots & \vdots \\ 0 & p_2 & \ddots & q_{n-1} & 0 \\ \vdots & \cdots & \ddots & 0 & q_n \\ 0 & \cdots & 0 & p_{n-1} & p_n \end{pmatrix}$$

This matrix form is called a tridiagonal matrix because the only nonzero elements occur in the diagonal or in the elements adjacent to the diagonal. A property of such a tridiagonal matrix is that for $0<p_i<1$, there are no repeated eigenvalues; in particular, there is a single eigenvalue of unity, and therefore there is a single stable probability distribution for a progressive drill. If a practice drill is performed for a large number of steps, then this unique distribution will be approached in the stochastic limit, and the statistical parameters associated with this distribution can be used to assess the player's performance at the drill.

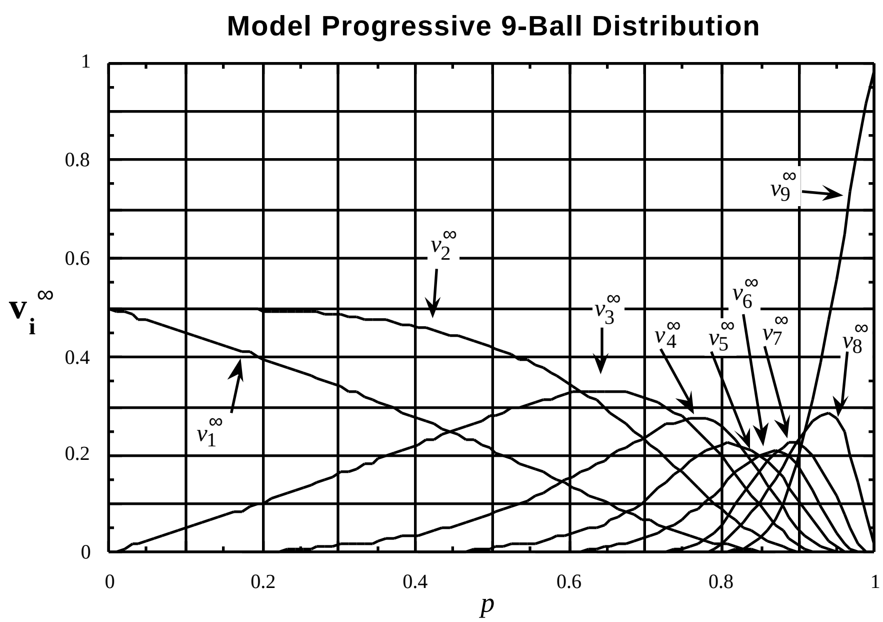


Fig. 5.8. The components of the stochastic limit distribution $\mathbf{v}^\infty$ are plotted as a function of the shot-success parameter p for the model progressive 9-ball drill with $p_i=p^{(i-1)}$. The distribution changes significantly as a function of p, and this means that the distribution is a sensitive measure of performance.

Problem 5.26: In the progressive 9-ball drill, the player starts by throwing the 9-ball randomly on the table, taking the cue ball in hand, and shooting the 9-ball. Upon success, the 8-ball and the 9-ball are thrown on the table and the player attempts to run both balls from ball in hand. In general, a successful run of i balls means that a run of i+1 balls is attempted, and a failure at a run of *i* balls means that, on the next turn, a run of *i*-1 balls is attempted. Assume that the probability of success for *i* balls is $p_i = p^{(i-1)}$ where *p* is an average probability of making an individual shot. What is the expected distribution for *p* equal to 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, and 0.99? What are the mean, median, mode, and standard deviations for each of these distributions?

Answer: The expected distributions are the stochastic limit $\mathbf{v}^\infty$ which is determined from the right eigenvector $\mathbf{R}_1$ of the matrix $\mathbf{M}^\infty$ associated with the eigenvalue $\lambda=1$, and scaled so that the elements total to unit probability. The coefficients of $\mathbf{R}_1$ are plotted as a function of the shot success parameter *p* in Fig. 5.8. These probability distributions for the specific values of *p* are given in Table 5.5, along with the associated statistical parameters. Because the distribution changes significantly with small changes in *p*, it provides a sensitive assessment of performance.

Table 5.5. Model Progressive 9-Ball Drill Statistics

p	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	$\bar{x}$	*Mode*	$\tilde{x}$	σ
.5	.210	.419	.280	.080	.011	.001	.000	.000	.000	2.264	2	2	.920
.6	.138	.346	.324	.149	.037	.005	.000	.000	.000	2.618	2	3	1.051
.7	.069	.231	.318	.237	.107	.031	.006	.001	.000	3.201	3	3	1.236
.8	.018	.088	.195	.256	.222	.135	.060	.020	.005	4.354	4	4	1.526
.9	.000	.004	.017	.052	.110	.177	.223	.227	.190	6.964	8	7	1.567
.95	.000	.000	.000	.002	.011	.038	.112	.272	.565	8.333	9	9	.928
.99	.000	.000	.000	.000	.000	.000	.005	.076	.918	8.912	9	9	.306

Exercise 5.1. Practice the progressive 9-ball drill over an extended period of time and accumulate the data for the number of successes and number of failures at each level. From these data, an empirical value of the success probability p_i at each level can be estimated. Use these empirical values and determine the corresponding stochastic limit distribution. Compare this hypothetical distribution to the actual distribution, which consists of the total attempts (successes+failures) at each level. If there are significant differences, then this shows where the most significant improvements in performance are possible. For example, if there seem to be too many small-*i* attempts, then additional focus may be needed for these "easy" cases, or there may be some intimidation on the long run attempts.

Almost any kind of shot or game situation may be turned into a progressive drill and subjected to this kind of stochastic analysis. For example, for 8-ball, the player might throw out an equal number of stripes and solids along with the 8-ball, take ball in hand, and attempt to run out. Upon success, one more ball of each type is thrown out at the beginning, until all 15 balls are initially on the table.

In the National Pool League (NPL) handicap system (see, for example, `http://www.accessone.com/~mavlon/handicap.html`), each player has a numerical skill rating estimate. If R_1 and R_2 are the skill ratings for two opponents, then the probability p that player-1 will win an individual game is assumed to be given by

$$p = \frac{1}{1 + 2^{-(R_1 - R_2)/30}}$$

or, equivalently, the rating difference between two opponents satisfies the relation

$$R_1 - R_2 = \frac{30}{\log 2} \log\left(\frac{p}{1-p}\right)$$

Skill ratings range from about 20 for beginners to around 80 for experienced amateur players to over 130 for professional-level players. Each additional rating difference of 30 points results in another factor of two in the ratio of game probabilities p/q. Matchups are chosen based on the analysis in P5.5. In general, for a given p_1, The match probability $W(p;m,n)$ is determined for values of $m+n$ that are reasonable for tournament play, and the combination that gives the match probability closest to W=0.5 is chosen. The following table contains four sets of matchups. Chart-8 is used for short matches when the time for each match needs to be minimized, Chart-10 is used for regular length matches, and Chart-12 is used when longer matches can be played. In some situations, short charts are used for lower-rated players and longer charts are used for higher-rated players. Longer and shorter charts than those shown here may also be used in particular league or tournament situations. Chart-20 is a very long match chart and is included for comparison purposes. When a player wins a match in the NPL system, his skill rating increases by a point, and when a player loses a match his skill rating decreases. Because of this adjustment, the skill rating estimate tends to fluctuate somewhat about a mean value that reflects the player's true skill rating. The skill rating value may be used to label the states in a game diagram, and because transitions are allowed only between nearby states, the game diagram for the NPL handicap system is the same as for a progressive drill as shown in Fig. 5.7, and the corresponding Markov transition matrix is tridiagonal.

In order to perform a stochastic simulation of the NPL handicap system, it is useful to introduce a few simplifying approximations. It is assumed that a particular player of interest, player-1, has a true skill that corresponds to a skill rating of R^{Actual}. He plays against an infinite number of opponents, all of whom have skill ratings that also correspond to $R^{Opponent}=R^{Actual}$. As player-1 plays against these opponents, his skill rating estimate will fluctuate about R^{Actual}. At any particular time player-1's apparent skill rating will be denoted $R^{Apparent}$. It is $R^{Apparent}$ that is used to determine the game matchup, using the charts in Table 5.6, but the actual game probability is determined by $R^{Actual}-R^{Opponent}=0$. For example, suppose that a tournament is using Chart-10, and the apparent rating difference is 5 points, which means that the matchup is 5:5. $W(0.5;5,5)=0.5$ then defines the transition probability for player-1 to advance to the next higher skill rating, and $(1-W(0.5;5,5))=0.5$ defines the probability for the player to fall back to the next lower skill rating. If player-1 wins this match, the apparent skill rating difference for his next match will be 6 points and the next matchup will be 5:4. From P5.6 it is seen that $W(0.5;5,4)=0.363$. Player-1 is now overrated and is more likely to lose this match than to win it. Similarly, when player-1 loses enough matches his apparent rating will be 6 points too low, the match probability will be $W(0.5;4,5)=0.637$. At this point, player-1 is underrated and is more likely to win than to lose. According to this mechanism, the player has a tendency to fluctuate about his true skill rating; if his apparent rating gets too low there is a tendency for him to start winning a majority of his matches and for his rating to adjust up back to its correct level, and if his apparent rating gets too high there is a tendency to lose a majority of his matches and for his rating to adjust back down to its correct level.

Table 5.6. Examples of four charts used in the NPL handicap system.

Chart-10

Rating Difference	Match Games
0-5	5:5
6-14	5:4
15-21	6:4
22-28	5:3
29-36	6:3
37-46	7:3
47-56	6:2
57-up	7:2

Chart-12

Rating Difference	Match Games
0-4	6:6
5-11	6:5
12-17	7:5
18-22	6:4
23-28	7:4
29-35	8:4
36-42	7:3
43-48	8:3
49-58	9:3
59-68	8:2
69-up	9:2

Chart-8

Rating Difference	Match Games
0-6	4:4
7-18	4:3
19-29	5:3
30-39	4:2
40-48	5:2
49-up	6:2

Chart-20

Rating Difference	Match Games
0-2	10:10
3-7	10:9
8-12	10:8
13-17	11:8
18-22	11:7
23-27	12:7
28-33	12:6
34-36	13:6
37-40	14:6
41-45	13:5
46-51	14:5
52-59	14:4
60-68	16:4
69-75	15:3
76-77	16:3
78-87	17:3
88-97	16:2
98-100	17:2
101-up	18:2

Fig. 5.9 shows the stochastic distributions $\mathbf{v}^{\infty}$ for the four charts in Table 5.6 with the above assumptions. These are called the natural distributions because they depend only on the granularity introduced by the matchups. In general, it is seen that there are

two components to the widths of a given distribution. One component is the flat region at the top which is due to the W=0.5 transition probability for near-zero apparent rating differences. This flat region is wider for the shorter-match charts than for the longer-match charts. The other component of the width is the falloff that is induced by the matchup differences. This component would occur even for longer matches than shown in Fig. 5.9, but in general the falloff is more rapid for longer matches than for shorter matches, as discussed in P5.7. The Chart-20 distribution displays both characteristics of the long-matches: a narrow flat region and a rapid falloff.

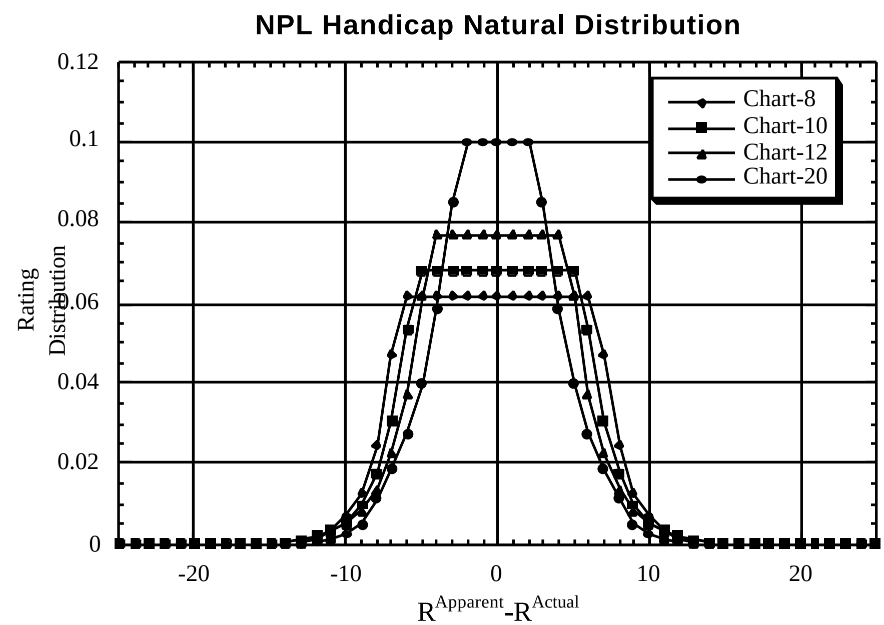


Fig. 5.9. The natural distributions $\mathbf{v}^{\infty}$ of rating variations are shown for four NPL charts. The variations away from the correct rating are relatively small in all cases, but the peak is sharper generally for longer matches than for shorter matches.

An improved simulation can be achieved by relaxing some of the simplifying approximations used in the preceding stochastic analysis. One approximation is that the opponents' skill ratings are estimated exactly. It can be assumed that there is a probability distribution $\{d_j\}$ of actual skill ratings for the opponents. Player-1 has an actual skill rating of R^{Actual} and an apparent skill rating of R_i ; the opponent has an actual skill rating of R_j , occurring with probability d_j, and an apparent skill rating of R^{Actual}. The probability for player-1 to win a match is then given by the expression

$$W_i = \sum_j d_j W\left(p_j; m_i, n_i\right)$$

in which p_j is determined by the actual skill difference (R^{Actual}-R_j) and the matchup m_i:n_i is determined by the apparent skill difference (R_i-R^{Actual}). This kind of expression involving summations of probability distributions is called a *convolution*. The question then arises as to what opponent distribution $\{d_j\}$ should be used. An obvious answer is to use the same distribution for both the opponents and for player-1. This distribution is determined in a self-consistent manner. Some reasonable approximation for $\{d_j\}$is assumed, and the corresponding stochastic distribution $\mathbf{v}^{\infty}$ for player-1 is determined from the eigenvalue analysis of the transition matrix. This stochastic distribution then defines a new $\{d_j\}$, which then results in a new transition matrix, which then results in a new stochastic distribution. After a few cycles of this process, the input distribution $\{d_j\}$ converges to the same as the output stochastic distribution $\mathbf{v}^{\infty}$, and self-consistency is achieved. This process in which the stochastic distribution depends on itself is called *autocorrelation*. In the case of the NPL handicap distributions, this has a very small effect, too small to notice the difference when plotted as in Fig. 5.9. The standard deviation for the Chart-8 distribution widens from 4.963 for the natural distribution to 4.978 with autocorrelation, for Chart-10 it widens from 4.634 to 4.654, for Chart-12 it widens from 4.269 to 4.291, and for Chart-20 it widens from 3.668 to 3.697. Further improvements in the simulation require additional assumptions about the distribution of actual and apparent skill ratings for the opponents, and about the day-to-day and match-to-match fluctuations of actual skill that all players display. In general, all of these effects tend to smooth and widen the stochastic distributions compared to the natural distributions shown in Fig. 5.9 and to the autocorrelated distributions described above. For the efficient numerical treatment of the convolution of several distribution variables, methods based on Fourier transforms are usually employed.

In addition to the matches of limited length that have been analyzed previously in this section, another common type of match is the *n-ahead* match. The players keep playing games until one of them manages to get *n* games ahead of the other player, and this terminates the match. It is also possible to handicap such a match, so that one of the players needs *m* games ahead to win, while the other player needs *n* games. The game diagram for a general handicapped *n*-ahead match is shown in Fig. 5.10. For a match of this type handicapped at *m:n*, the game graph has (m+n+1) nodes, *m* of which are on one side of the starting node *S*0, and *n* of which are on the other. It is assumed that the probability of winning an individual game is independent of the score, and for simplicity it is assumed that the breaker of each game does not affect the game probability, although it is straightforward to incorporate differing probabilities for these situations if such data is available. The Markov transition matrix for the *n*-ahead game always has two eigenvalues of unity; this may be verified by expanding the secular equation in cofactors and minors first along the first column (corresponding to the losing node *L*), and then along the last column (corresponding to the winning node *W*), exposing two $(1-\lambda)$ factors in the characteristic polynomial. The following problem shows these general features for a specific game diagram, but the general approach can be applied to any *n*-ahead type match situation.

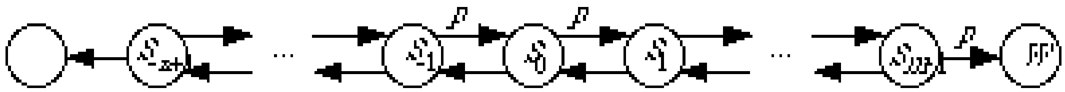


Fig. 5.10. The game diagram for a general handicapped *n*-ahead type match is shown. From the starting node S_0, player-1 needs *m* games to win the match and player-2 needs *n* games.

Problem 5.27: Two players play a handicapped 3-ahead type match. Compute the player-1 match probability *W* as a function of *p*, the probability of winning an individual game, if the match is handicapped at 1:5, 2:4, 3:3, 4:2, and 5:1.

Answer: There are 7 nodes in the game diagrams for all of these cases, and the Markov transition matrix **M** for all of these cases is given by

$$\mathbf{M} = \begin{pmatrix} 1 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & p & 0 & q & 0 & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1 \end{pmatrix}$$

The different handicaps correspond to different choices for the initial probability. The 1:5 match corresponds to the vector $\mathbf{v}^0=(0,0,0,0,0,1,0)^T$, the 2:4 match corresponds to $\mathbf{v}^0=(0,0,0,0,1,0,0)^T$, the 3:3 match corresponds to $\mathbf{v}^0=(0,0,0,1,0,0,0)^T$, the 4:2 match corresponds to $\mathbf{v}^0=(0,0,1,0,0,0,0)^T$, and the 5:1 match corresponds to $\mathbf{v}^0=(0,1,0,0,0,0,0)^T$. The eigenvectors and eigenvalues of this matrix can be determined in closed form. The stochastic limit for this match situation is determined from

$$\mathbf{M}^\infty = \begin{pmatrix} 1 & \frac{(1-pq)(1-3pq)-p^5}{(1-pq)(1-3pq)} & \frac{(1-pq)(1-3pq)-p^4}{(1-pq)(1-3pq)} & \frac{q^3}{(1-3pq)} & \frac{q^4}{(1-pq)(1-3pq)} & \frac{q^5}{(1-pq)(1-3pq)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{p^5}{(1-pq)(1-3pq)} & \frac{p^4}{(1-pq)(1-3pq)} & \frac{p^3}{(1-3pq)} & \frac{(1-pq)(1-3pq)-q^4}{(1-pq)(1-3pq)} & \frac{(1-pq)(1-3pq)-q^5}{(1-pq)(1-3pq)} & 1 \end{pmatrix}$$

from which it is seen that the match probability for the various cases are given by relatively simple ratios of polynomials. These probabilities are plotted as a function of *p* for the various match situations in Fig. 5.11. Note that Fig. 5.11 could have been determined numerically even if closed-form expressions for $\mathbf{M}^\infty$ were not available.

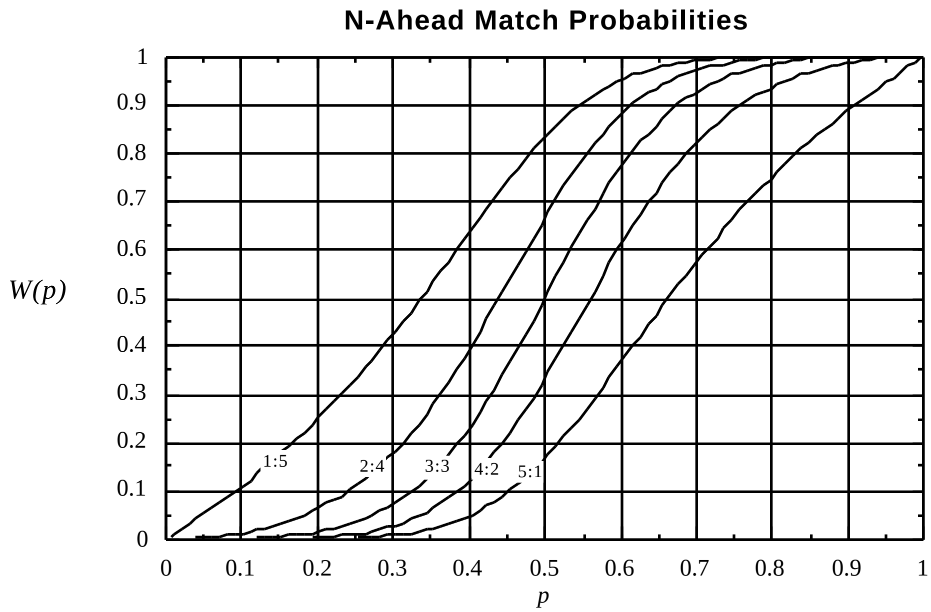


Fig. 5.11. Match probability *W* as a function of the player-1 game probability *p* for *n*-ahead matches handicapped at 1:5, 2:4, 3:3, 4:2, and 5:1.

Acknowledgments: Much of this material has been compiled over a long period of time. The author first became interested in the physics of pool during a college physics course (not an uncommon occurrence). Some more recent material has been added as a result of ongoing discussions in the newsgroup `rec.sport.billiard` involving many participants. This newsgroup is highly recommended to anyone interested in discussions involving the various aspects of pool and billiards games.

Further Reading: Considering that many important and interesting aspects of pool and billiards may be understood with only simple application of classical physics, and that quite useful results can be obtained even with rather crude approximations, there has been traditionally relatively little physics included in most instructional pool books. Simple physics problems involving pool balls are often included in problem sets in physics text books, but these are not discussed usually in the context of using the results in actual play, but rather as a device to teach a physical principle or in the application of an analytic method. Some of the exceptions to this trend are the regular columns by Bob Jewett in *Billiards Digest*. Another good publication is the book "*The Physics of Pocket Billiards*" by W. C. Marlow. While this present manuscript concentrates mostly on theoretical relations combined with practice exercises, Marlow's book includes descriptions of experimental setups to measure tip-ball contact times, ball-ball contact times, various coefficients of friction, and many other interesting things.