In the special case of V_{cpy} =0, then also V_{by}=0 and Δ_{total} vanishes, indicating that the departure angle of the cue ball and object ball is exactly a right angle.

The initial velocity of the cue ball immediately after collision is given by $\mathbf{V}_c = \left(V_0 \sin(\alpha) - V_{by}\right)\hat{\mathbf{j}}$. The *magnitude* of this velocity depends on the object ball throw, but its *direction* is independent of any frictional forces. If the cue ball has no spin about the horizontal axis (*i.e.* only sidespin, no backspin or topspin), then this initial direction is unchanged by the sliding friction of the cloth. The cue ball will slow down upon achieving natural roll, but the velocity direction will remain unchanged. In this sense, the trajectory of the cue ball after the collision is less dependent on the ball-ball coefficient of friction μ_{bb} than the object ball trajectory. This observation is useful in judging and executing accurate stun shot caroms.

Exercise 4.1: Experiment with stun shot caroms. Begin by placing the cue ball a few inches away from the object ball, and cueing exactly in the center. The cue ball should not curve after the collision. Mark the position of the cue ball center at the collision point and the two contact points where the balls touch the cushions. Measure the angle and determine how close is the departure angle to a right angle. Include shots with sidespin to determine the effects of Δ_{total} on the departure angle. With some practice, stun shot caroms can be executed very accurately. Stun shot caroms are particularly useful in 9-ball.

Problem 4.12: Determine the total angular momentum immediately before and after the collision relative to the point that corresponds to the cue ball center at the moment of collision. Is angular momentum conserved? (ignore the linear velocity components due to the vertical frictional forces)

Answer: There are two contributions to the total angular momentum. One is the rotational contributions of the balls spinning about their centers, $\mathbf{L}^{spin} = \mathbf{I}\,\omega$, and the other is the orbital contribution of the centers of mass moving about the point of origin, $\mathbf{L}^{orbit} = \mathbf{r} \times \mathbf{p}$. Before the collision, these contributions are

$$\mathbf{L}_0^{orbit} = \mathbf{r}_0(t) \times \mathbf{p}_0(t) = \left(\mathbf{V}_0 t\right) \times \left(M\mathbf{V}_0\right) = 0$$

$$\mathbf{L}_0^{spin} = \mathbf{I}\,\omega_0$$

$$\mathbf{L}_0 = \mathbf{L}_0^{orbit} + \mathbf{L}_0^{spin} = \mathbf{I}\,\omega_0$$

After the collision the contributions are

$$\begin{aligned}\mathbf{L}_b^{orbit} &= \mathbf{r}_b(t) \times \mathbf{p}_b(t) = \left(2R\hat{\mathbf{i}} + \mathbf{V}_b t\right) \times \left(M\mathbf{V}_b\right) \\ &= 2MR\left(\hat{\mathbf{i}} \times \mathbf{V}_b\right) = 2MR\left(\hat{\mathbf{i}} \times \left(V_{bx}\hat{\mathbf{i}} + V_{by}\hat{\mathbf{j}}\right)\right) = 2MRV_{by}\hat{\mathbf{k}} \\ &= -2I\omega_{bz}\hat{\mathbf{k}}\end{aligned}$$

$$\mathbf{L}_b^{spin} = \mathbf{I}\,\omega_b$$

$$\mathbf{L}_c^{orbit} = \mathbf{r}_c(t) \times \mathbf{p}_c(t) = (\mathbf{V}_b t) \times (M\mathbf{V}_b) = 0$$

$$\mathbf{L}_c^{spin} = \mathbf{I}\,\omega_c = \mathbf{I}(\omega_0 + \omega_b)$$

$$\mathbf{L} = \mathbf{L}_c^{orbit} + \mathbf{L}_c^{spin} + \mathbf{L}_b^{orbit} + \mathbf{L}_b^{spin} = \mathbf{I}\,\omega_0 + 2I\omega_{by}\hat{\mathbf{j}}$$

The total angular momentum difference before and after the collision is then

$$\mathbf{L} - \mathbf{L}_0 = 2I\omega_{by}\hat{\mathbf{j}}$$

The total angular momentum is always conserved except for the horizontal component about the *y*-axis, which is conserved only when ω_{by}=0. This component arises from the vertical frictional force during the collision, and vanishes only when ω'_{0y}=0 (*i.e.* for stun shot collisions). The vertical component of angular momentum is always conserved, as is the other horizontal component about the *x*-axis; the orbital angular momentum arising from the object ball throw compensates exactly for the change in the spin angular momentum. This compensation cannot occur for the vertical frictional force because of the constraint of the table surface. In the above equations, the vertical linear acceleration was neglected, but even if it had been included for the jumped ball (as determined in P4.9), the corresponding contribution from the nonjumped ball during the collision is eliminated by the table surface. Indeed, as discussed previously, because the vertical components of linear momentum are not conserved in the collision, it should not be expected that the angular momentum components due to these same frictional forces could be conserved using the same simple analysis.

In the previous few problems, various aspects of object ball throw have been examined. The object ball throw affects the trajectories of the balls immediately after the collision. The behavior of the balls after the collision is determined by both the initial post-collision conditions of the balls and by the action of the cloth friction on the sliding balls which was discussed in some detail in the previous sections. The results of the present section heretofore, involving ball-ball interaction will now be combined with the results of the previous sections to examine the behavior of the sliding balls as a function of the collision conditions, and eventually, as a function of the tip-ball contact point. In the following discussions, object ball throw will be largely ignored in order to simplify the derivations. In most cases, the effects of object ball throw may be included, at the cost of some additional complexity, but this adds relatively little to the basic understanding of the situations. The first situation to be considered is the behavior of a natural roll cue ball after collision with an object ball. This special case is particularly central to pool and billiards because of the special importance of natural roll.

Problem 4.13: What is the angle of deflection of a natural roll cue ball as a function of the object ball cut angle after the collision and after natural roll is achieved by both balls?

(ignore friction between the balls)

Answer: With no ball-ball friction, the initial cue ball deflection direction is $\pi/2$ (90 degrees) from the object ball cut angle. In terms of unit vectors $\hat{\mathbf{i}}'$ and $\hat{\mathbf{j}}'$ in the x' and y' coordinate directions respectively in Fig. 4.2, the initial velocity vectors immediately after collision are given by

$$\mathbf{V}_b = V_0 \cos(\alpha)\left(\cos(\alpha)\hat{\mathbf{i}}' - \sin(\alpha)\hat{\mathbf{j}}'\right)$$

$$\mathbf{V}_c = V_0 \sin(\alpha)\left(\sin(\alpha)\hat{\mathbf{i}}' + \cos(\alpha)\hat{\mathbf{j}}'\right) \quad .$$

The cut angle α is the angle between vectors $\mathbf{V}_b$ and $\mathbf{V}_0$. There is no initial object ball angular velocity immediately after the collision, so only the speed changes and not the direction upon achieving natural roll. The final natural roll velocity is given by

$$\mathbf{V}_{b,NR} = \tfrac{5}{7}\mathbf{V}_b = \tfrac{5}{7}V_0 \cos(\alpha)\left(\cos(\alpha)\hat{\mathbf{i}}' + \sin(\alpha)\hat{\mathbf{j}}'\right) \ .$$

The situation is somewhat different for the cue ball. The cue ball has natural roll before the collision, $V_0 = R\omega'_{0y}$, and this angular velocity is unchanged by the collision with the object ball. The ball-cloth friction from this initial angular velocity creates a force component in the $\hat{\mathbf{i}}'$ direction only. The final velocity vector for the cue ball is

$$\mathbf{V}_{c,NR} = \tfrac{5}{7}\mathbf{V}_c + \tfrac{2}{7}V_0\hat{\mathbf{i}}' = \left(\tfrac{5}{7}V_0 \sin^2(\alpha) + \tfrac{2}{7}V_0\right)\hat{\mathbf{i}}' + \left(\tfrac{5}{7}V_0 \sin(\alpha)\cos(\alpha)\right)\hat{\mathbf{j}}' \ .$$

The cue ball deflection angle θ, relative to the velocity vector $\mathbf{V}_0$, after natural roll is achieved, is determined by

$$\tan(\theta) = \frac{\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha) + \frac{2}{5}}$$

Immediately after the collision, the cue ball path is a parabola as determined in P2.3. The frictional force accelerates the cue ball until natural roll is achieved. At the point that natural roll is achieved, the cue ball rolls in a straight line with no acceleration. The angle between this straight line and the initial velocity direction $\mathbf{V}_0$ is the deflection angle θ which satisfies the above equation.

Problem 4.14: Show that $\tan(\alpha + \theta) = \tfrac{7}{2}\tan(\alpha)$

Answer: Using the tangent addition relation $\tan(\alpha + \theta) = \dfrac{\tan(\alpha) + \tan(\theta)}{1 - \tan(\alpha)\tan(\theta)}$ with $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\theta) = \dfrac{\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha) + \frac{2}{5}}$ gives

$$\tan(\alpha + \theta) = \frac{\sin(\alpha)\left(\sin^2(\alpha) + \cos^2(\alpha) + \frac{2}{5}\right)}{\frac{2}{5}\cos(\alpha)} = \tfrac{7}{2}\tan(\alpha)$$

Problem 4.15: What cut angle α maximizes the natural roll deflection angle θ?

Answer: Rewrite the above expression as $\theta = \arctan\left(\tfrac{7}{2}\tan(\alpha)\right) - \alpha$. Differentiate with respect to α to obtain

$$\frac{d\theta}{d\alpha} = \frac{14}{4 + 45\sin^2(\alpha)} - 1.$$

Setting the derivative to zero and solving for α gives

$$\alpha_{(\theta_{max})} = \arcsin\left(\frac{\sqrt{2}}{3}\right) = 0.49088 = \frac{\pi}{6.3999} \quad [= 28.125 \text{ deg}]$$

Note that this is just a bit thicker than a half-ball hit, which is a $\pi/6$ or a 30 degree cut angle (neglecting collision induced throw).

Problem 4.16: What is the maximum deflection angle θ for a natural-roll cue ball collision?

Answer: Substitution of $\alpha_{(\theta_{max})}$ gives

$$\theta_{max} = \arctan\left(\tfrac{7}{2}\tan\left(\alpha_{(\theta_{max})}\right)\right) - \alpha_{(\theta_{max})}$$

$$= \frac{\pi}{2} - 2\alpha_{(\theta_{max})} = 0.58903 = \frac{\pi}{5.3335} \quad [= 33.749 \text{ deg}]$$

This is very useful to know because a natural-roll cue ball carom at this angle is intrinsically more accurate than a cut shot with the same cut angle as demonstrated in the following problem.

Problem 4.17: If the object ball is cut about 2 degrees away from that corresponding to the maximum deflection angle as determined in P4.15, what is the change in the cue ball deflection angle?

Answer: If the cut angle is 2 degrees less, then

$$\theta = \arctan\left(\tfrac{7}{2}\tan(26\text{deg})\right) - 26\text{deg} = 33.64\text{deg}$$

which is 0.11 degrees away from the maximal value as determined in P4.16. If the cut angle is 2 degrees more, corresponding to a half-ball hit of 30 degrees, then

$$\theta = \arctan\left(\tfrac{7}{2}\tan(30\text{deg})\right) - 30\text{deg} = 33.67\text{deg}$$

which is 0.08 degrees away from the maximal value. In both cases, the cue ball deflection angle is much more stable to small deviations than the object ball cut angle.

Problem 4.18: What is the relation between the cut angle α and the natural roll deflection angle θ for small cut angles α?

Answer: For small angles (measured in radians), $\tan(x) \approx x$. The relation, $\tan(\alpha + \theta) = \tfrac{7}{2}\tan(\alpha)$, from P4.14 then gives

$$\theta \approx \tfrac{5}{2}\alpha \qquad \text{[for small } \alpha\text{]}.$$

This relation is useful to know when playing position using natural roll on nearly straight-in shots. It is difficult to achieve a larger amount of topspin than $V_0 = R\omega_0$ with a direct cue-tip/cue-ball shot due to the risk of miscue (see P1.7). However, higher spin/speed ratios can be achieved with carom shots. A higher spin/speed ratio would result in a smaller factor than that in the above equation.

Problem 4.19: What is the cut angle α at which exactly half of the kinetic energy of a natural-roll cue ball is transferred to the object ball? What is the corresponding natural roll deflection angle θ? At this angle, what are the final kinetic energies of both balls?

Answer: When the cue ball has natural roll, $V_0=R\omega_0$, the total kinetic energy is

$$T = \tfrac{1}{2} M V_0^2 + \tfrac{1}{2} I\omega_0^2 = \tfrac{7}{10} M V_0^2$$

The energy of the object ball immediately after collision is

$$T_b = \tfrac{1}{2} M V_b^2 = \tfrac{1}{2} M V_0^2 \cos^2(\alpha)$$

Setting $T_b=\tfrac{1}{2}T$ and simplifying gives

$$\alpha_{(\frac{1}{2}T)} = \arccos\left(\sqrt{\tfrac{7}{10}}\right) = 0.57964 = \frac{\pi}{5.4199} \quad [= 33.211 \text{ deg}]$$

This angle is unchanged as the object ball achieves natural roll. The corresponding deflection angle after natural roll of the cue ball is achieved is

$$\theta_{(\frac{1}{2}T)} = \arctan\left(\tfrac{7}{2}\tan(\alpha_{(\frac{1}{2}T)})\right) - \alpha_{(\frac{1}{2}T)} = 2\alpha_{(\frac{1}{2}T)} - \alpha_{(\frac{1}{2}T)} = \alpha_{(\frac{1}{2}T)}$$

The relation $\tfrac{7}{2}\tan(\alpha_{(\frac{1}{2}T)}) = \tan\left(2\alpha_{(\frac{1}{2}T)}\right)$, used to simplify the above expression, may be verified using the tangent addition formula in P4.14. Therefore, when the final deflection angles are equal for both balls, then each ball has the same kinetic energy immediately after the collision. Note that the cut angle at which this occurs is just a bit thinner than that for a half-ball hit (which would be 30 degrees, neglecting collision induced throw).

The final object ball and cue ball kinetic energies, using $V_{b,NR}$ and $V_{c,NR}$ from P4.13 are

$$T_{b,NR} = \tfrac{1}{2} M V_{b,NR}^2 = T_0\left(\tfrac{25}{49}\cos^2(\alpha)\right)$$

$$T_{c,NR} = \tfrac{1}{2} M V_{c,NR}^2 = T_0\left(\tfrac{25}{49}\sin^4(\alpha) + \tfrac{20}{49}\sin^2(\alpha) + \tfrac{4}{49} + \tfrac{25}{49}\sin^4(\alpha)\cos^2(\alpha)\right)$$

where T_0 is the initial cue ball translational energy. These relations are satisfied for any cut angle α. Substitution of $\cos^2(\alpha_{(\frac{1}{2}T)})={}^7/_{10}$ and $\sin^2(\alpha_{(\frac{1}{2}T)})={}^3/_{10}$ for the specific half-energy cut angle results in

$$T_{b,NR} = T_{c,NR} = \tfrac{5}{14} T_0 \ .$$

Not only is the energy divided equally between the two balls upon collision with a cut angle of $\alpha_{(\frac{1}{2}T)}$, but the final energies of the two balls are equal after both balls achieve natural roll. The distance that a ball rolls after achieving natural roll, neglecting subsequent cushion and ball collisions, is directly proportional to the natural roll kinetic energy. This relation is useful in situations in which it is necessary that both the object ball and the cue ball roll the same distance, and as a point of reference when unequal distances are required.

In Fig. 4.7 the deflection angle θ of a natural roll cue ball, as determined in P4.13 and P4.14, is plotted as a function of the object ball cut angle α. Also shown on the same graph is the derivative curve $\left(\frac{d\theta_{NR}}{d\alpha}\right)$ as determined in P4.14. The points on this curve corresponding to a half-ball hit, the maximum deflection angle $\alpha_{(\theta_{NR:\max})}$ from P4.14, and the deflection angle corresponding to splitting the kinetic energy as determined in P4.19, are also plotted. The derivative curve is monotonic in the range shown in Fig. 4.7 (in general, it is an even function, symmetric about α=0). The derivative curve starts with a value of $5/2$ at α=0 (see P4.15), decreases to the value of zero at $\alpha_{(\theta_{\max})}$, and then approaches its asymptotic value of $-5/7$ as the cut angle approaches π/2. Another point of interest shown in Fig. 4.7 is the value of the cut angle α at which the slope $\left(\frac{d\theta_{NR}}{d\alpha}\right)$ has a value of one. This occurs at $\alpha_{crit}=\arcsin\left(\sqrt{1/15}\right)=.26116$ [=14.963 deg]. For cut angles less than α_{crit}, $\left|\frac{d\theta_{NR}}{d\alpha}\right|>1$ and the natural roll cue ball trajectory is more sensitive than the object ball trajectory to small variations in the cut angle. However for the rest of the range of cut angles, $\left|\frac{d\theta_{NR}}{d\alpha}\right|<1$ and the cue ball trajectory is less sensitive than the object ball trajectory. Less sensitivity means that it is easier for the shooter to control, and this may be used to advantage, for example, in placing the cue ball more precisely in position and safety play.

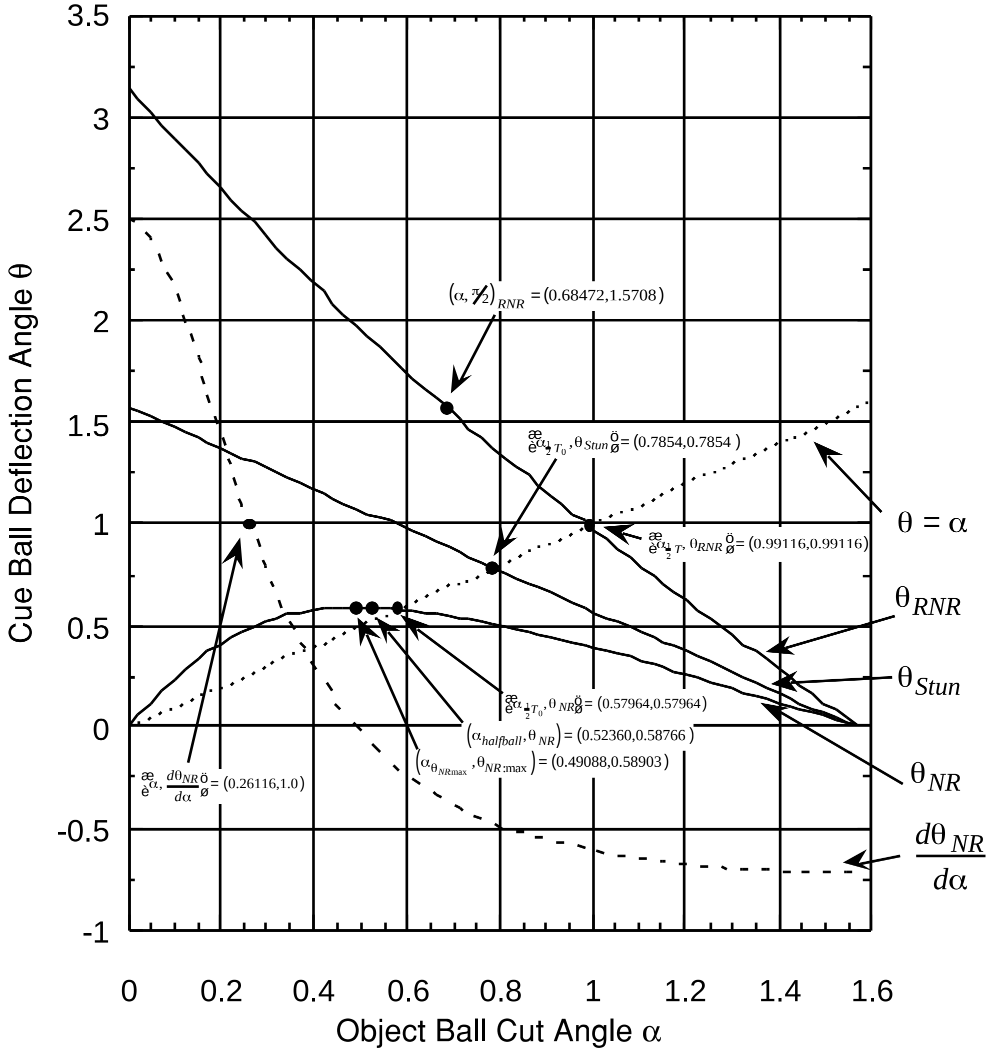

Fig. 4.7. The post-collision natural roll cue ball deflection angle is shown as a function of the object ball cut angle. The θ_{NR} curve is applicable when the cue ball has natural roll before the collision. θ_{Stun} is when the cue ball has no spin before the collision. θ_{RNR} is when the cue ball has reverse natural roll before the collision. The straight line $\theta=\alpha$ corresponds to an equal splitting of the kinetic energy after both balls achieve natural roll. Also shown is the dashed curve defined by $\left(\frac{d\theta_{NR}}{d\alpha}\right)$. Several important individual points on each of these curves are also shown as discussed in the text.

Problem 4.20: If the cue ball is not rotating upon impact with the object ball (a stun shot), at what cut angle α is half of the kinetic energy transferred? What are the final energies of the balls? (neglect any frictional forces between the balls)

Answer: Taking the velocities immediately after the collision from P4.13, the initial kinetic energies are

$$T_b = \tfrac{1}{2} M V_b^2 = \tfrac{1}{2} M V_0^2 \cos^2(\alpha)$$

$$T_c = \tfrac{1}{2} M V_c^2 = \tfrac{1}{2} M V_0^2 \sin^2(\alpha)$$

Equating these two energies gives

$$\tan^2(\alpha) = 1$$

$$\alpha = \arctan(1) = \frac{\pi}{4} \quad [= 45 \text{ deg}]$$

Each ball has initially after the collision an energy of $\tfrac{1}{2}T_0$. Since neither ball has any angular velocity immediately after the collision, both balls slow down upon achieving natural roll by $\tfrac{5}{7}$ of the initial ball velocities. There is no change of angle, since the velocity directions of the balls do not change. The natural roll kinetic energy of each ball is then $(\tfrac{1}{2})(\tfrac{5}{7})^2 T_0 = (\tfrac{25}{98}) T_0$. Compared to the results of P4.19 involving natural roll of the cue ball, it is seen that the cut angle is thinner and that the final energies of both balls are smaller relative to T_0 with a stun shot than with natural roll. This half-energy cut angle point for stun shots is shown on the θ_{Stun} curve in Fig. 4.7. The θ_{Stun} curve is a straight line that ranges from the limiting values of $\theta_{Stun}=\pi/2$, at cut angle $\alpha=0$, to $\theta_{Stun}=0$, at $\alpha=\pi/2$.

Problem 4.21: What is the natural roll cue ball deflection angle as a function of the cue ball spin ω_{0y} at the moment of collision and the object ball cut angle?

Answer: Generalizing the results of P4.13, it is convenient to write the natural roll cue ball velocity in terms of the spin/speed ratio $J_{0y}.=(R\omega_{0y}/V_0)$.

$$\mathbf{V}_{c,NR} = \tfrac{5}{7}\mathbf{V}_c + \tfrac{2}{7} V_0 J_{0y}\hat{\mathbf{i}}' = \tfrac{5}{7} V_0 \left(\sin^2(\alpha) + \tfrac{2}{5} J_{0y}\right)\hat{\mathbf{i}}' + \tfrac{5}{7} V_0 \left(\sin(\alpha)\cos(\alpha)\right)\hat{\mathbf{j}}'$$

The cue ball deflection angle is determined by the ratio of the two components.

$$\tan(\theta) = \frac{\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha) + \tfrac{2}{5} J_{0y}}$$

Using the tangent addition relation, this may be written as

$$\tan(\alpha + \theta) = \left(\frac{1 + \tfrac{2}{5} J_{0y}}{\tfrac{2}{5} J_{0y}}\right)\tan(\alpha)$$

For the natural roll condition, J_{0y}=+1, these results all agree with those of P4.13-P4.14.

Problem 4.22: In P4.19 and P4.20 it is seen that a particular cut angle splits evenly both the initial kinetic energy and the natural roll kinetic energies of the two balls. Under what

conditions will a cut angle split both energies? (assume ω_z=0)
Answer: Half of the initial kinetic energy is transferred when $T_b=\frac{1}{2}T_0$. This occurs when

$$\cos^2(\alpha) = \left(\tfrac{1}{2} + \tfrac{1}{5} J_{0y}^2\right)$$

where J_{0y} is the spin/speed ratio ($R\omega_{0y}/V_0$). The natural roll kinetic energy is split evenly when $T_{b,NR}=T_{c,NR}$. Using the previous natural roll conditions, this occurs when

$$\cos^2(\alpha_{NR}) = \left(\tfrac{1}{2} + \tfrac{1}{5} J_{0y}\right)$$

The angles α and α_{NR} are equal only when

$$J_{0y}(J_{0y} - 1) = 0$$

There are only two possible solutions to this equation: J_{0y} =1, the natural roll situation discussed in P4.19, and J_{0y} =0, the stun shot condition discussed in P4.20. For other spin/speed ratios, there will be one angle α that splits the initial kinetic energy, and a separate angle α_{NR} that splits evenly the natural roll kinetic energies.

Problem 4.23: If the cue ball has reverse natural roll (RNR), $V_0=-R\omega'_{0y}$, what is the relation between the cut angle α and the natural roll deflection angle θ?
Answer: For reverse natural roll, J_{0y} =-1. Referring to the result in P4.21,

$$\tan(\alpha + \theta) = -\tfrac{3}{2}\tan(\alpha)$$

The sign factor in this equation indicates that ($\alpha+\theta$) is in a different quadrant than α. Specifically, $0\le\alpha\le\pi/2$ is always in the first quadrant, and $\pi/2\le(\theta+\alpha)\le\pi$ is always in the second quadrant. Taking the appropriate quadrant for θ gives the relation

$$\theta = \arctan\left(-\tfrac{3}{2}\tan(\alpha)\right) - \alpha + \pi \quad .$$

For small cut angle α, it is seen that

$$\theta \approx \pi - \tfrac{5}{2}\alpha \qquad \text{[for small } \alpha\text{]}$$

The same factor of $\frac{5}{2}$ is seen for the RNR draw shot as for the (topspin) natural roll shot in P4.15. However, in the case of a draw shot the deviation is away from the reverse direction π (or 180 degrees), rather than the forward direction. As in the case with topspin, it is difficult to achieve a larger amount of draw than $V_0=-R\omega_0$ with a normal direct cue-tip/cue-ball shot due to the risk of miscue (see P1.7). However, higher spin/speed ratios can be achieved with carom and masse shots.

Problem 4.24: In P4.19 and P4.20 it is seen that the kinetic energy of the cue ball and object ball is split evenly when the cut angle is equal to the cue ball deflection angle for J_{0y}=1 and J_{0y}=0. Show that this condition is true for arbitrary J_{0y}. What is the cut angle that splits the natural roll energy of a reverse natural roll collision? How does this angle compare to the natural roll angle from P4.19.
Answer: From P4.22, the post-collision natural roll kinetic energy is split evenly when $\cos^2(\alpha) = \left(\tfrac{1}{2} + \tfrac{1}{5} J_{0y}\right)$ and $\sin^2(\alpha) = \left(\tfrac{1}{2} - \tfrac{1}{5} J_{0y}\right)$. Substitution of these relations into the general deflection angle equation of P4.21 gives

$$\tan(\theta) = \frac{\sin(\alpha)\cos(\alpha)}{\sin^2(\alpha) + \frac{2}{5} J_{0y}} = \frac{\sin(\alpha)\cos(\alpha)}{\cos^2(\alpha)}$$

$$= \tan(\alpha)$$

or in general $\theta = \alpha$ when the natural roll kinetic energy is split evenly. This line is shown in Fig. 4.7. The above equation for the cut angle may be written as

$$\alpha = \theta = \arcsin\left(\sqrt{\tfrac{1}{2} - \tfrac{1}{5} J_{0y}}\right)$$

In particular, for reverse natural roll, J_{0y}=-1, the half-energy cut angle is given by

$$\alpha_{\frac{1}{2}T,RNR} = \arcsin\left(\sqrt{\tfrac{7}{10}}\right) = 0.99116 = \frac{\pi}{3.1696} \quad [= 56.789\text{deg}]$$

From comparison with P4.19, it is seen that $\alpha_{\frac{1}{2}T,RNR} + \alpha_{\frac{1}{2}T,NR} = \frac{\pi}{2}$. This is an example of the general relation

$$\alpha_{\frac{1}{2}T,J_{0y}} + \alpha_{\frac{1}{2}T,-J_{0y}} = \frac{\pi}{2}$$

which follows from the relation, $\cos^2(\alpha_{\frac{1}{2}T,J_{0y}}) = \sin^2(\alpha_{\frac{1}{2}T,-J_{0y}})$

The reverse natural roll deflection angle is shown as a function of the object ball cut angle in Fig. 4.7. Considering θ_{RNR} as a function of cut angle α, it is seen that θ_{RNR} ranges from zero, for very thin cuts, to π, for very thick cuts. In contrast θ_{NR} from P4.15 only ranged from zero to a bit over $\pi/6$. Since natural roll topspin and reverse natural roll backspin represent the practical extremes of cue ball spin (neglecting collision effects and masse), the area between the θ_{NR} and θ_{RNR} curves in Fig. 4.7 respresents all possible practically allowed shots. The area between the θ_{Stun} curve and the θ_{RNR} curve represents all possible draw shots, and the area between the θ_{Stun} and θ_{NR} curves represents all possible topspin shots. Inspection shows that the area associated with draw shots is much larger than that associated with topspin shots. This means that there is much more flexibility with respect to carom angles with draw than with topspin, and correspondingly, that topspin shots are usually less sensitive than draw shots to variations in the cut angle or amount of spin. It may be seen in Fig. 4.7 that θ_{RNR} is almost a straight line, with an average slope of about twice that of θ_{Stun}. Since θ_{Stun} is relatively easy to determine, this allows in turn θ_{RNR} to be estimated for any cut angle simply by multiplying θ_{Stun} by 2. Inspection of Fig. 4.7 shows that this simple factor will always overestimate the actual deflection angle. The following problem demonstrates the magnitude of error of this approximation.

Problem 4.25: At what cut angle does a reverse natural roll cue ball deflect at exactly a right angle?

Answer: From P4.23, the desired cut angle satisfies the relation

$$\tan(\alpha \quad + \tfrac{\pi}{2}) = -\tfrac{3}{2}\tan(\alpha \quad)$$

Using the identity $\tan(\alpha + \frac{\pi}{2}) = -1/\tan(\alpha)$, α may be determined to be

$$\alpha = \arctan\left(\sqrt{\tfrac{2}{3}}\right) = 0.68472 = \frac{\pi}{4.5881} \quad [= 39.232\text{deg}]$$

This point is plotted on the θ_{RNR} curve in Fig. 4.7. The simple "factor of 2" estimate from the stun-shot curve would have predicted this angle to be $\pi/4$ (or 45 degrees), which would have been about 12% in error. The correct cut angle α is about midway between a half-ball cut angle and the $\pi/4$ angle.

Problem 4.26: For a given cut angle α, what sidespin/speed ratio will result in no horizontal tangential frictional forces?

Answer: The surfaces of the balls must not slide against each other in order for the frictional forces to vanish during the collision. The velocity of the cue ball contact point just before the collision is the sum of the linear velocity $\mathbf{V}_0$ and the instantaneous velocity due to the angular velocity about the vertical axis $\omega \times \mathbf{r}$. The contact point velocity is given by

$$\begin{aligned}\mathbf{V}_{cp} &= V_0 \cos(\alpha)\hat{\mathbf{i}} + \left(V_0 \sin(\alpha) + R\omega_{0z}\right)\hat{\mathbf{j}} - R\omega'_{0y}\cos(\alpha)\hat{\mathbf{k}} \\ &= V_{cpx}\hat{\mathbf{i}} + V_{cpy}\hat{\mathbf{j}} + V_{cpz}\hat{\mathbf{k}}\end{aligned}$$

When V_{cpy}=0, then the horizontal frictional forces vanish. Solving for the ratio $R\omega_{0z}/V_0$ gives

$$J_{0z} = \frac{R\omega_{0z}}{V_0} = -\sin(\alpha)$$

Problem 4.27: Using the initial spin/speed ratio and the final natural roll spin/speed ratio from P3.6, and the V_{cpy}=0 relation from P4.26, what cue tip contact points will result in no horizontal tangential frictional forces between the two colliding balls with a cut angle α?

Answer: For the spin/speed ratio immediately after cue tip contact, the contact points are given by the vertical line satisfying

$$\sin(\alpha) = \frac{5y'_{tip}}{2R}$$

Note that the object ball contact point satisfies the relation, $y'_{cp}=-R\sin(\alpha)$. This gives the relation between y'_{tip} and y'_{cp} as

$$y'_{tip} = -\tfrac{2}{5}\, y'_{cp}$$

The sign difference means that the cue tip impact parameter is in the opposite hemisphere from the object ball contact point. Note that in the limit of an extreme cut shot of angle $\pi/2$, this result agrees with that of P3.5; that is, "sideways natural roll" is achieved with a horizontal impact parameter of $^2/_5R$. This relation is useful when the object ball collision occurs very soon after the cue tip contact, before the friction between the ball and cloth has time to change the velocity.

When the cue ball is allowed to achieve natural roll before colliding with the object ball, the desired cue tip contact points satisfy

$$\sin(\alpha) = \frac{7}{2}\left(\frac{y'_{tip}}{z_{tip}}\right)$$

$$y'_{tip} = -\left(\frac{2y'_{cp}}{7R}\right) z_{tip}$$

For a given cut angle α, this is a straight line that passes through the origin (0,0). An easy way to estimate the sets of points defined by this straight line is as follows. Refer to Fig. 4.8. Determine the correct contact point at height $z_{tip}=\frac{7}{5}R$. At this contact point, natural roll would be achieved immediately (see P3.5), and the natural roll horizontal offset is the same as the initial horizontal offset determined above, namely the contact point would be $(y,'z)=(-\frac{2}{5}y'_{cp},\frac{7}{5}R)$. The set of desired points is then given by drawing a straight line between this particular contact point and the point at the very bottom of the ball (0,0). In particular, the point on this straight line that is the minimum distance from the center is on the small circle as shown in P3.7.

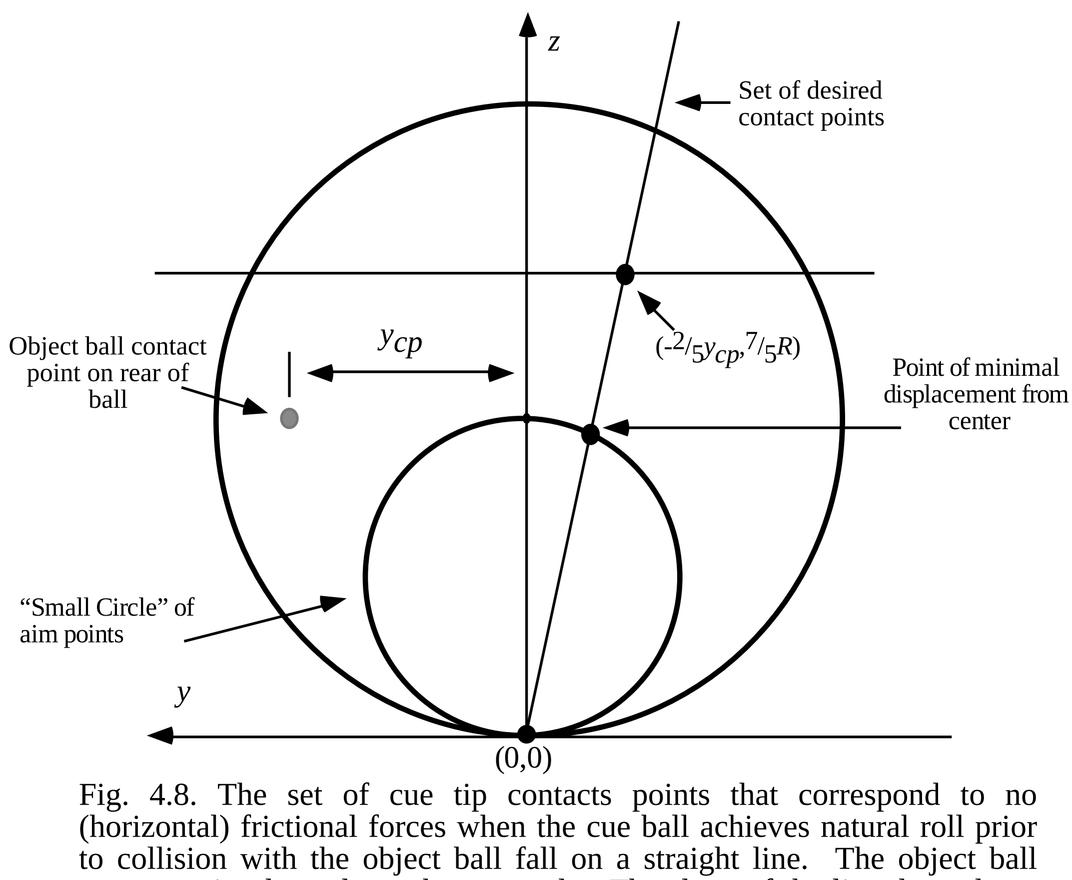

Fig. 4.8. The set of cue tip contacts points that correspond to no (horizontal) frictional forces when the cue ball achieves natural roll prior to collision with the object ball fall on a straight line. The object ball contact point depends on the cut angle. The slope of the line depends on the object ball contact point y'_{cp} as indicated.

5. Statistics

The mathematical fields of statistical analysis, combinatorial analysis, stochastic analysis, and game theory are all useful in both physics and pool, and they are all interesting fields of study for the amateur. Statistical methods, which is used here in a general way to include all of these fields, can be used to assess performance, to judge a technique or strategy, and to predict future outcomes based on previous and perhaps incomplete information. These and other uses of statistics will be examined in this section. First some elementary background material and notation will be introduced.

The *average*, or *arithmetic mean*, of a set of values $\{x_i\}$, called a *population*, is

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

N is the number of values and the index i runs over the members of the set. There can be repetitions among the values x_i, and it may be more convenient to sum over the distinct values $\{y_i\}$, weighted by their repetitions $\{n_i\}$, rather than the individual members of the sample space. In this case the mean may be written as

$$\bar{x} = \sum_{i=1}^{N_{val}} n_i y_i \Big/ \sum_{i=1}^{N_{val}} n_i \quad \text{where } N = \sum_{i=1}^{N_{val}} n_i$$

The probability for each distinct value is

$$p_i = \frac{n_i}{N}$$

and these form a set of nonnegative numbers $\{p_i\}$; this gives another useful expression for the mean.

$$\bar{x} = \sum_{i=1}^{N_{val}} p_i y_i$$

Note that the mean does not necessarily correspond to a member of the sample set.

The set of probabilities $\{p_i\}$ and the corresponding distinct values $\{y_i\}$ defines the probability distribution. For many purposes, it is convenient to consider the probability as a function of the value, $p(y)$. For an ordered set of values $\{y_i\}$, say with $y_i < y_{i+1}$, and corresponding probabilities $\{p_i\}$, there is a cumulative probability defined by

$$p_m^{cum} = \sum_{i=1}^{m} p_i = p_{m-1}^{cum} + p_m$$

The cumulative probability increases monotonically to its maximum value of 1. It is sometimes useful to study properties of various subsets of the population, and the cumulative probability is often used to pick out, for example, the bottom third, or the middle third, or the top quartile, or the top 5%.

Another useful property of a distribution is the *median*. Suppose that the individual members of the sample space x_i are ordered by value. The median of the sample is the value of the INT((N+1)/2) element in the ordered list, where INT() implies truncation to an integer value. (There are several conventions used to handle the situation in which there is an even number of members, and the two middle members have different values; for simplicity, this situation will not be considered in this section.) In terms of the ordered probability distribution, the median is determined by the smallest value m which satisfies

$$p_m^{cum} \geq \frac{1}{2}$$

The median of the set $\{x_i\}$ is denoted $\tilde{x}$. If the members of the sample space are chosen randomly, then it is just as likely that a value less than the median will be picked as a value that is greater than the median.

The *distribution maximum* or *mode* is the value corresponding to the largest probability value. For a given set, a maximum may not exist, or it may not be unique. If the distribution is symmetric and centrally peaked, then the mode, the median, and the mean will all be the same. If the probability distribution is symmetric, but not necessarily peaked, then the median and the mean are the same but the mode may be different. If the distribution is skewed, meaning that it is not peaked about a central value, then the mean, the median, and the mode will generally have all different values.

Another important value that characterizes a sample set is the standard deviation, which, like the mean, may be computed in various ways in terms of the sample elements, repetition counts, and probability distributions.

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2} = \sqrt{\frac{1}{N}\sum_{i=1}^{N_{val}} n_i\left(y_i - \bar{x}\right)^2} = \sqrt{\sum_{i=1}^{N_{val}} p_i\left(y_i - \bar{x}\right)^2}$$

$$= \sqrt{\left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right) - \bar{x}^2} = \sqrt{\left(\frac{1}{N}\sum_{i=1}^{N_{val}} n_i y_i^2\right) - \bar{x}^2} = \sqrt{\left(\sum_{i=1}^{N_{val}} p_i y_i^2\right) - \bar{x}^2}$$

If the sample values are very tightly clustered about the mean value, then σ will be small, and if the sample values are broadly spread apart then σ will be large. The *variance* is defined as the quantity σ^2.

Problem 5.1: Given the set $\{x\}=\{0,1,1,4,5\}$, compute the mean using all three methods, the median, the mode, and the standard deviation. What are these same quantities for the set $\{x\}=\{0,1,1,8,9\}$.

Answer: For the first set, the distinct values and corresponding probabilities are $\{y_i\}=\{0,1,4,5\}$ and $\{p_i\}=\{1/5, 2/5, 1/5, 1/5\}$. The mean may be written as

$$\bar{x} = \frac{0+1+1+4+5}{5} = \frac{0+2\cdot 1+4+5}{1+2+1+1} = \tfrac{1}{5}\cdot 0 + \tfrac{2}{5}\cdot 1 + \tfrac{1}{5}\cdot 4 + \tfrac{1}{5}\cdot 5 = \tfrac{11}{5}$$

The median is the value of the third element (i.e. (5+1)/2) in this ordered list, $\tilde{x}=x_3=1$. The largest distribution value is $p_2=2/5$, so the mode is $y_2=1$. In this case the mode and the median happen to have the same value, but they both differ from the mean. The standard deviation of the first set is

$$\sigma = \sqrt{\frac{0^2 + 2\ 1^2 + 4^2 + 5^2}{5} - \left(\frac{11}{5}\right)^2} = \sqrt{\frac{94}{25}} = 1.939$$

For the second set the mean is

$$\bar{x} = \tfrac{1}{5}\ 0 + \tfrac{2}{5}\ 1 + \tfrac{1}{5}\ 8 + \tfrac{1}{5}\ 9 = \tfrac{19}{5}$$

and the standard deviation is σ=Sqrt(374/25)=3.868. For this set, the median is still $\tilde{x}=x_3=1$, and the distribution maximum $p_2=2/5$ still occurs for $y_2=1$, the same as for the first set. For both sets, the mean value does not correspond to a set member. The standard deviation is larger for the second set than for the first set, reflecting the wider range of values.

It is sometimes useful to merge various subsets of values into one large set. If the subset size, mean, and standard deviation is known for each of the subsets, then it is possible to compute the size, mean, and standard deviation of the combined set without knowing the individual values. The parameters of the combined set are given by

$$N = \sum_i N_i$$

$$\bar{x} = \frac{1}{N}\sum_i N_i \bar{x}_i$$

$$\sigma^2 = \frac{1}{N}\sum_i N_i \sigma_i^2 + \frac{1}{N}\sum_i N_i \left(\bar{x}_i - \bar{x}\right)^2$$

The summations in these equations are over the subsets, not the individual elements. The combined average is simply the weighted average of the subset averages. The variance of the combined set contains two contributions, the first is the weighted mean of the subset variances, and the second is the weighted variance of the subset means.

Problem 5.2: Compute the mean and variance for the combined set {0,0,1,1,1,1,4,5,8,9}={0,1,1,4,5} $\cup$ {0,1,1,8,9} using the results from P5.1.

Answer: N=5+5=10

$\bar{x}$=(5(11/5)+5(19/5))/10=3

σ^2=(5(94/25)+5(374/25)+5(11/5-3)^2+5(19/5-3)^2)/10=10

It may be verified that these values agree with those computed using the individual elements of the combined set.

In some situations the *sample space* is only a subset of a larger *population space*. The sample space may be used to estimate the statistical parameters (mean, median, mode, standard deviation, etc.) of the population space, or the population space statistics may be used to predict possible subspace statistics. In some cases the population space may be too large to handle, or may even be infinite in size, in which case only a smaller sample space is available. There are two conceptual ways of constructing a sample space. One way is by randomly choosing elements from the population space, and setting aside the member once it has been chosen so that it cannot be drawn again; the other way is to replace the elements as they are chosen so that they may be chosen again. Some care must be taken with this choice to ensure that the sample space gives the best possible representation of the population space.

Suppose a population space consists of *N* distinguishable objects (e.g. numbered slips of paper). If one member of this set is chosen, and if the probability for all the members is the same (e.g. the slips are the same size and mixed well before selection), then there are *N* possible, equally likely, outcomes. Now consider choosing two members of the set, without replacement. What is the number of possible outcomes? The act of drawing two objects can be thought of conceptually in two steps: drawing one object, setting it aside, and then drawing the second object. There are *N* possible outcomes after the first draw, and (*N*-1) possible outcomes for the second draw, so it would appear that there might be *N*(*N*-1) possible outcomes. However, if the order of drawing the two objects is unimportant, then this overcounts the outcomes by a factor of two, and the correct answer would be *N*(*N*-1)/2. In the general case, what is the number of possible outcomes for choosing *m* distinguishable objects where the order that they might be drawn is unimportant? The answer is the binomial coefficient which is written

$$\binom{N}{m} = \frac{N!}{m!(N-m)!} = \frac{N(N-1)(N-2)\cdots(N-m+1)}{1 \cdot 2 \cdot 3 \cdots m}$$

The numerator in the last expression is the number of ways to select the *m* objects one at a time, without replacement, from the population space, and the denominator is the number of permutations of these objects to account for the fact that their order is irrelevant. The binomial coefficient $\binom{N}{m}$ is often pronounced "*N* choose *m*" to stress this important relationship. Binomial coefficients satisfy the recursion

$$\binom{N}{m} = \binom{N-1}{m-1} + \binom{N-1}{m}$$

with the boundary conditions $\binom{N}{0} = \binom{N}{N} = 1$. This leads to "Pascal's Triangle"

```
                1
             1     1
          1     2     1
       1     3     3     1
    1     4     6     4     1
 1     5    10    10     5     1
1    6    15    20    15     6    1
…                 …                …
```

in which the row is determined by *N* and the element within the row corresponds to *m*. In the triangle, each element is the sum of the two nearby elements in the row above it, a result of the two-term recursion. The name "binomial coefficient" comes from the fact that these numbers are the coefficients of the individual elements in the term-by-term expansion of $(p+q)^n$.

$$(p+q)^n = \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} = q^n + npq^{n-1} + \ldots + np^{n-1}q + p^n$$

Problem 5.3: Given the sample set {1,2,3,4}, enumerate all of the ways of choosing zero, one, two, three, and four elements without replacement.
Answer: There is 1 way to choose zero elements: {}; there are 4 different ways to choose one element: {1}, {2}, {3}, and {4}; there are 6 ways to choose two elements: {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, and {3,4}; there are 4 ways to choose three elements: {1,2,3}, {1,2,4}, {1,3,4}, and {2,3,4}; there is 1 way to choose four elements: {1,2,3,4}. These numbers, 1, 4, 6, 4, and 1, agree with the n=4 row of Pascal's triangle and with the closed-form expression for the binomial coefficients.

Suppose that the probability for a "successful" event to occur is p. The probability for a failure is $q=(1-p)$. If the sample space is infinite, or if the space is finite and the sampling is done with replacements, then the probability for success does not change upon repetition and the probability for two consecutive successes is p^2. The probability for a single success and a single failure is $2pq$, because there are two ways to arrive at this result, each of which has probability pq. The probability of two failures is q^2. In the general case, the probability of obtaining m successes and n failures after $N=n+m$ attempts is given by $P(p;m,n)=\binom{m+n}{m}p^m q^n$. Comparison with the binomial expansion shows that this probability is the m^{th} term in the expansion of $(p+q)^{(m+n)}$; consequently such distributions are called *binomial distributions*.

Problem 5.4: Two players are playing 9-ball, and the probability that player-1 will win an individual game is $p=2/3$. What is the probability that after 4 games the score will be 3:1? Enumerate all the possible ways of arriving at this game score.
Answer: Using the above equation the probability is

$$P(2/3;3,1)=\binom{4}{1}\left(\tfrac{2}{3}\right)^3\left(\tfrac{1}{3}\right)^1 = 4\left(\tfrac{8}{27}\right)\left(\tfrac{1}{3}\right) = \tfrac{32}{81} = 0.395$$

There are four ways of arriving at this game score: LWWW, WLWW, WWLW, and WWWL. The probability of each of these individual ways occurring is $p^3q^1 = \left(\tfrac{2}{3}\right)^3\left(\tfrac{1}{3}\right)^1 = 0.0988$.

Problem 5.5: Two players are playing 9-ball, and the probability that player-1 will win an individual game is $p=2/3$. The match is handicapped at 3:2, meaning that player-1 must win 3 games whereas player-2 must win only 2 games in order to win the match. What is the probability that player-1 will win this match? If the match is handicapped at N_1:N_2, what is the general expression that player-1 will win?

Answer: There are two ways that player-1 can win the match handicapped at 3:2, namely 3:0, and 3:1. In order to arrive at a 3:0 score, player-1 must win the last game from a 2:0 score; the probability for this to occur is $pP(p;2,0)=\left(\frac{2}{3}\right)\binom{2}{0}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^0 = \frac{8}{27} = 0.296$. In order to arrive at a 3:1 score, player-1 must win the last game from a 2:1 score; the probability for this to occur is $pP(p;2,1)=\left(\frac{2}{3}\right)\binom{3}{1}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{8}{27} = 0.296$. The probability of player-1 winning the match is the sum of these two terms, W=16/27=0.593.

In the general case, there are N_2 ways for player-1 to win the match: N_1:0, N_1:1, ..., N_1:(N_2-1). The probability for the individual N_1:m case is $pP(p;N_1-1,m)$. The probability that player-1 will win the match is the sum over all of the individual probabilities

$$W(p;N_1,N_2) = \sum_{m=0}^{N_2-1} pP\left(p;N_1-1,m\right) = \sum_{m=0}^{N_2-1} \binom{N_1+m-1}{m} p^{N_1} q^m$$

In the following two problems, the match probability W is examined, first with the game probability p fixed and varying the matchup N_1 and N_2, and then with N_1 and N_2 fixed and varying p.

Problem 5.6: Write a computer program to compute a table of values containing $P(p;m,n)$ for $0 \le m \le 9$ and $0 \le n \le 9$. From this table, compute the corresponding $W(p;m,n)$ tables for $1 \le m \le 10$ and $1 \le n \le 10$. Compute these tables for $p=\frac{1}{2}$, for which the players are equally likely to win an individual game, and for $p=\frac{2}{3}$, for which player-1 is twice as likely to win an individual game as player-2.

Answer: For this purpose, it is better to formulate the $P(p;m,n)$ table construction using the following recursion approach (which is similar to computing Pascal's triangle).

$P(p;0,0)=1$

$P(p;0,n)=qP(p;0,n-1)$; for n=1,2,...9

$P(p;m,0)=pP(p;m-1,0)$; for m=1,2,...9

$P(p;m,n)=qP(p;m,n-1)+pP(p;m-1,n)$; for m=2,...,9 and for n=2,...,9

Basically, this recognizes the fact that to arrive at a score of m:n, either player-1 must win the last game from a score of (m-1):n, which occurs with probability p, or player-2 must win from a score of m:(n-1), which occurs with a probability q.

The $W(p;m,n)$ table is then constructed in a similar manner.

$W(p;m,1)=pP(p;m-1,0)$; for m=1,...,10

$W(p;m,n)=W(p;m,n-1)+pP(p;m-1,n-1)$; for m=1,...,10 and for n=2,...,10

These tables are included below for the two specified values. Note that the $P(\frac{1}{2};m,n)$ table is symmetric (i.e. $P(\frac{1}{2};m,n)=P(\frac{1}{2};n,m)$), as would be expected for two equally matched players. Note also that the $W(2/3;3,2)$ entry agrees with the hand-calculated value from P5.5.

Such a program may be easily written in almost any programming language or spreadsheet. It is sometimes handy to have such a program available when directing tournaments, or even for personal use, in order to determine fair handicapped matchups between players of varying strengths.

Problem 5.7. Compare the $W(p;n,n)$ and the $W(p;2n,n)$ match probabilities for $1 \leq n \leq 10$ numerically as a function of the game probability p.

Answer: Using the computer program from P5.6, the appropriate elements of the W table may be determined as a function of p. These match probabilities are shown in Fig. 5.1. In general, it may be observed that each curve of $W(p;m,n)$ is an increasing function of the game probability p. It is seen that $W(1/2;n,n)=1/2$ for all matches. This means that if player-1 is the stronger player, $p>1/2$, it is to his advantage to play a longer even matchup rather than a shorter match, but if player-1 is the weaker player, $p<1/2$, then it is to his advantage to play a shorter match. A beginner might be able to win a game (i.e. a 1:1 match) against a professional, but it is most unlikely that he would win a longer 10:10 match.

There is no single common point of exact intersection for the $W(p;2n,n)$ curves; these curves cross at slightly different values of p. If a match probability of $W=1/2$ is defined as "fair", then it is clear in Fig. 5.1 that player-1 must have a larger game probability p to survive a 2:1 match than a 4:2 match. An interesting region occurs for the 2:1 and 4:2 curves after they intersect ($W(.641;2,1)=W(.641;4,2)=.411$) but before the point corresponding to $W(.686;4,2)=1/2$. In this domain, $.641<p<.686$, $W<1/2$ for both curves, so player-1 is expected to lose both matches, yet it is still to his advantage to play the longer match. This handicapped situation is in contrast to the even-matchup situation in which the expected winner always benefits from the longer match. Such a domain exists for the other pairs of $2n$:n matchup curves, but it becomes much smaller because the curves are steeper for longer matches. Furthermore, in the domain $.5<p<.641$, before the 2:1 and 4:2 curves intersect, player-1 is the stronger player but his best chances of winning are with the shorter 2:1 match. Again, this is in contrast to the even-matchup situation in which the stronger player always benefited the most with longer matches.

P(½;*m*,*n*)

m\n	0	1	2	3	4	5	6	7	8	9
0	1.000	0.500	0.250	0.125	0.063	0.031	0.016	0.008	0.004	0.002
1	0.500	0.500	0.375	0.250	0.156	0.094	0.055	0.031	0.018	0.010
2	0.250	0.375	0.375	0.313	0.234	0.164	0.109	0.070	0.044	0.027
3	0.125	0.250	0.313	0.313	0.273	0.219	0.164	0.117	0.081	0.054
4	0.063	0.156	0.234	0.273	0.273	0.246	0.205	0.161	0.121	0.087
5	0.031	0.094	0.164	0.219	0.246	0.246	0.226	0.193	0.157	0.122
6	0.016	0.055	0.109	0.164	0.205	0.226	0.226	0.209	0.183	0.153
7	0.008	0.031	0.070	0.117	0.161	0.193	0.209	0.209	0.196	0.175
8	0.004	0.018	0.044	0.081	0.121	0.157	0.183	0.196	0.196	0.185
9	0.002	0.010	0.027	0.054	0.087	0.122	0.153	0.175	0.185	0.185

W(½;*m*,*n*)

m\n	1	2	3	4	5	6	7	8	9	10
1	0.500	0.750	0.875	0.938	0.969	0.984	0.992	0.996	0.998	0.999
2	0.250	0.500	0.688	0.813	0.891	0.938	0.965	0.980	0.989	0.994
3	0.125	0.313	0.500	0.656	0.773	0.855	0.910	0.945	0.967	0.981
4	0.063	0.188	0.344	0.500	0.637	0.746	0.828	0.887	0.927	0.954
5	0.031	0.109	0.227	0.363	0.500	0.623	0.726	0.806	0.867	0.910
6	0.016	0.063	0.145	0.254	0.377	0.500	0.613	0.709	0.788	0.849
7	0.008	0.035	0.090	0.172	0.274	0.387	0.500	0.605	0.696	0.773
8	0.004	0.020	0.055	0.113	0.194	0.291	0.395	0.500	0.598	0.685
9	0.002	0.011	0.033	0.073	0.133	0.212	0.304	0.402	0.500	0.593
10	0.001	0.006	0.019	0.046	0.090	0.151	0.227	0.315	0.407	0.500

P(⅔;*m*,*n*)

m\n	0	1	2	3	4	5	6	7	8	9
0	1.000	0.333	0.111	0.037	0.012	0.004	0.001	5E-04	2E-04	5E-05
1	0.667	0.444	0.222	0.099	0.041	0.016	0.006	0.002	9E-04	3E-04
2	0.444	0.444	0.296	0.165	0.082	0.038	0.017	0.007	0.003	0.001
3	0.296	0.395	0.329	0.219	0.128	0.068	0.034	0.016	0.007	0.003
4	0.198	0.329	0.329	0.256	0.171	0.102	0.057	0.030	0.015	0.007
5	0.132	0.263	0.307	0.273	0.205	0.137	0.083	0.048	0.026	0.013
6	0.088	0.205	0.273	0.273	0.228	0.167	0.111	0.069	0.040	0.022
7	0.059	0.156	0.234	0.260	0.238	0.191	0.138	0.092	0.057	0.034
8	0.039	0.117	0.195	0.238	0.238	0.207	0.161	0.115	0.077	0.048
9	0.026	0.087	0.159	0.212	0.230	0.214	0.179	0.136	0.096	0.064

W(⅔;*m*,*n*)

m\n	1	2	3	4	5	6	7	8	9	10
1	0.667	0.889	0.963	0.988	0.996	0.999	1.000	1.000	1.000	1.000
2	0.444	0.741	0.889	0.955	0.982	0.993	0.997	0.999	1.000	1.000
3	0.296	0.593	0.790	0.900	0.955	0.980	0.992	0.997	0.999	0.999
4	0.198	0.461	0.680	0.827	0.912	0.958	0.980	0.991	0.996	0.998
5	0.132	0.351	0.571	0.741	0.855	0.923	0.961	0.981	0.991	0.996
6	0.088	0.263	0.468	0.650	0.787	0.878	0.934	0.965	0.983	0.991
7	0.059	0.195	0.377	0.559	0.711	0.822	0.896	0.942	0.969	0.984
8	0.039	0.143	0.299	0.473	0.632	0.759	0.851	0.912	0.950	0.973
9	0.026	0.104	0.234	0.393	0.552	0.690	0.797	0.873	0.925	0.957
10	0.017	0.075	0.181	0.322	0.476	0.618	0.737	0.828	0.892	0.935

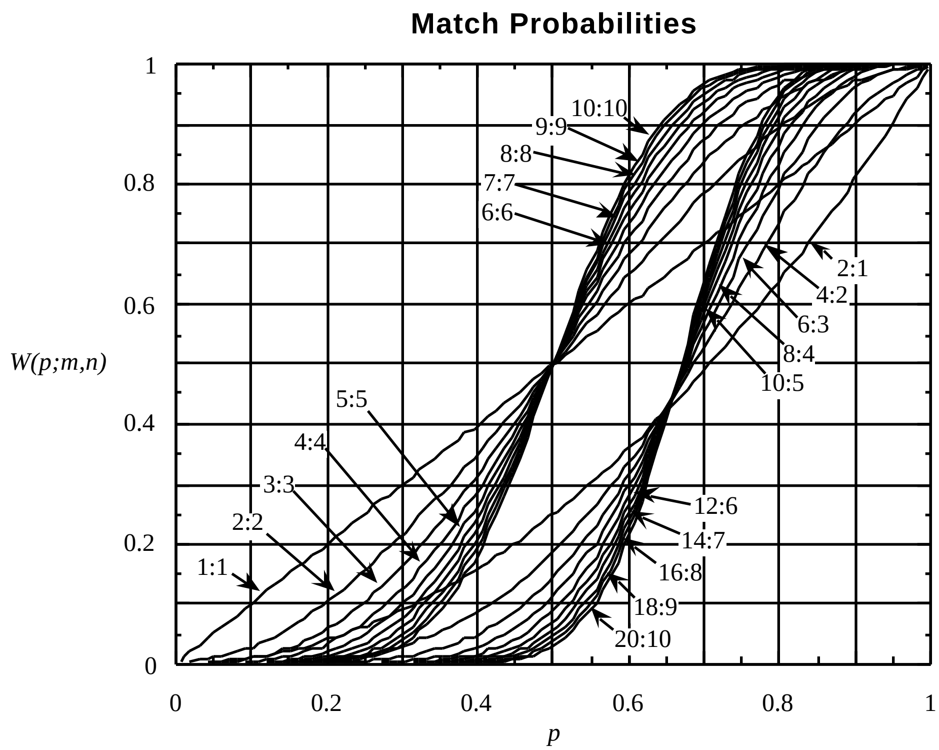

Fig. 5.1. The match probability *W* as a function of the individual game probability *p* for even *n:n* matchups and uneven 2*n:n* matchups of various lengths. For all of the individual curves, the match probability is an increasing function of the game probability. The steepness of a curve is related to how sensitive is the match outcome to the game probability.

Problem 5.8. A strong player is negotiating a matchup with a weaker opponent and he knows that his game probability against this opponent is $p=2/3$. He is offered a choice between a single long match of 9:5, and a 3:1 match of sets where each set is handicapped at 3:3. Which option is best for player-1?

Answer: At first this seems very complicated, so it is best to break the problem down into smaller pieces that are easier to understand. Player-1 will win the long match with a probability of $W(2/3;9,5)=.552$ according to the table in P5.6. He will win a 3:3 set with a probability of $W(2/3;3,3)=.790$, also according to the table in P5.6. The match probability for the second option is given by $W(.790;3,1)$. That is, the statistical analysis for winning multiple-set matches is the same as that for winning multiple-game matches, but with the variable p being the set probability instead of the game probability. This may be computed using the program in P5.6, or from the polynomial expression from P5.5: $W(p;3,1)=p^3$. In either case, the result is seen to be $W(.790;3,1)=.493$. Player-1 would have a small 5.2% advantage over player-2 in the long-match format, but he would have a slight 0.7% disadvantage with this particular set format.

In either case, player-1 must win 9 games total in order to win the match. In the long match format, player-2 needs to win 5 games to win the match, whereas in the set format he needs only to win 3 games, provided they are all in the same set. In the set format, player-2 can win as many as 6 games and still lose the match, provided they are split evenly with 2 games in each set. There are apparently no shortcuts, based simply on the total games required by each player, that will give the correct choice in these negotiations. The actual statistical analysis is required to correctly assess each possible option.

In the general case of multiple-set matches, the player-1 match probability is given by the general expression

$$W^{match} = W\left(W\left(p;N_1^{set},N_2^{set}\right);N_1^{match},N_2^{match}\right)$$

in which the required games per set and sets per match are indicated and in which p is the individual game probability for player-1.

For a given N with $N=m+n$, binomial distributions characterized by the probabilities $P(p;m,n)$ are centrally peaked for $p\approx\frac{1}{2}$ (i.e. the peak occurs near $m\approx N/2$), the peak is shifted toward large m values for large $p\approx1$, and the peak is shifted toward small m values for small $p\approx0$. Because binomial distributions are so common, it is useful to characterize the peak $\tilde{x}$, the mean $\bar{x}$, and the standard deviation σ in a general way.

Problem 5.9: Compute the mode, the mean, and the standard deviation of a binomial distribution in terms of N and p.

Answer: The possible values of a binomial distribution correspond to the integers $\{m\ ;m=0,\ldots N\}$ and the corresponding probabilities are given by $P(p;m,N\text{-}m)$. The mode, or distribution peak, occurs for the smallest value of m for which $P(p;m+1,N\text{-}m\text{-}1)<P(p;m,N\text{-}m)$. The peak of a binomial distribution is given by

$$\tilde{x} = m_{small} = Ceiling(\ Np - q\ )$$

where $Ceiling(x)$ denotes the smallest integer that is greater than or equal to x. The mean is given by

$$\begin{aligned}\bar{x} &= \sum_{m=0}^{N} mP(p;m,N-m) = \sum_{m=0}^{N} m\frac{N!}{m!(N-m)!}p^m q^{N-m}\\ &= \sum_{m=0}^{N-1}\frac{N(N-1)!}{m!(N-1-m)!}p^{m+1}q^{N-1-m} = Np\sum_{m=0}^{N-1}\frac{(N-1)!}{m!(N-1-m)!}p^m q^{N-1-m}\\ &= Np(p+q)^{N-1} = Np\end{aligned}$$

The mean and the mode of a binomial distribution differ by, at most, one. A similar sequence of operations gives the standard deviation of a binomial distribution.

$$\sigma = \sqrt{Npq}$$

An important property of the binomial distribution is that for large *N*, it approaches the *normal distribution* defined by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2}$$

This distribution is symmetric about the mean and is peaked at the mean. It is often useful to shift and scale the domain of the distribution using the equation $z = (x-\bar{x})/\sigma$. In terms of these dimensionless transformed values, called *standard units*, the normal distribution takes the simple form

$$P(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

In this *standard form*, the normal distribution is peaked at z=0 and has a standard deviation of σ=1. Areas under the normal distribution correspond to various cumulative probabilities. However, the form of the normal distribution does not allow for a simple closed-form expression of the antiderivative, so integrals must be computed numerically or interpolated from tables. One form for these tables is in terms of the symmetric integral $P^{cum}(z_c) = \int_{-z_c}^{z_c} P(z)\,dz$. The following short table gives some of the more commonly used cumulative probabilities and their corresponding critical values z_c.

Table 5.1. Normal Distribution Critical Values

$P^{cum}(z_c)$	.9973	.99	.98	.96	.9545	.95	.90	.80	.6827	.50
z_c	3	2.58	2.33	2.05	2	1.96	1.645	1.28	1	.6745

Problem 5.10: When two players play 9-ball the probability that player-1 will win any particular game is 0.52. These players play 120 games. What is the expected mean score for player-1 and the expected variation about this mean score? What is the range of scores that would be expected to occur 95% of the time? What is the range expected to occur 50% of the time?

Answer: The possible game scores form a binomial distribution. The mean score for player-1 is $\bar{x}=Np$=120(0.52)=62.4. The standard deviation is $\sigma=\sqrt{Npq}=\sqrt{120(.52)(.48)}$=5.47. For 120 games, the binomial distribution can be approximated by a normal distribution. The critical value corresponding to 95% is z_c=1.96. $z_c\sigma$=(1.96)(5.47)=10.7, so there is approximately a 95% probability that the final game score for player-1 will be between $\bar{x}-z_c\sigma$=51 and $\bar{x}+z_c\sigma$=73. There is only about a 5% chance that the final score will be outside of this range. For the 50% range, $z_c\sigma$=(.6745)(5.47)=3.69, so there is a 50% chance that the player-1 game score will be between 59 and 66 using the normal approximation to the binomial distribution. The exact probability, using the exact binomial statistics as in P5.6, for this range of scores is 53.5%, which shows that the normal distribution approximation is quite reliable.

Suppose that the probability p corresponds to some average probability of successfully executing a shot, and runlength statistics are of interest, where "runlength" means the number of consecutive successful shots. The chances of success on the first shot is p, and for two consecutive successful shots is p^2, and so on. The probability of running n balls or greater is p^n. What is the probability of running exactly n balls and then missing? The answer is $r_n=p^n q$ where $q=(1-p)$. The set $\{r_n\}$ then defines a probability distribution for a population of infinite size.

Problem 5.11: What is the mode, mean, and median runlength for a probability distribution defined by $r_n=p^n q$ as a function of p?

Answer: The ratio of two successive runlength probabilities, $r_{n+1}/r_n=p<1$ shows that the distribution is monotonically decreasing, and therefore the maximum of the distribution occurs always at n=0, regardless of p. This shows that the mode is not particularly useful for predicting typical outcomes if the distribution is severely skewed. The mean runlength is

$$\bar{r} = \sum_{n=0}^{\infty} n r_n = (1-p)\sum_{n=1}^{\infty} n p^n$$

That is, a run of length n occurs with probability r_n. It may be verified by straightforward division that

$$\frac{1}{(1-p)} = \sum_{n=0}^{\infty} p^n = 1 + p + p^2 + \ldots + p^k + \ldots$$

Differentiating both sides with respect to p , followed by multiplication by p, gives the identity

$$\frac{p}{(1-p)^2} = \sum_{n=1}^{\infty} n p^n = p + 2p^2 + 3p^3 + \ldots + kp^k + \ldots$$

Substitution of this relation into the expression for the mean runlength gives

$$\bar{r} = \frac{p}{(1-p)} = \frac{p}{q}$$

$$p = \frac{\bar{r}}{(\bar{r}+1)}$$

The first equation gives the average runlength in terms of the individual shot probability, whereas the second gives the individual shot probability as a function of the average runlength.

The median runlength is the smallest value m that satisfies the equation

$$\tfrac{1}{2} \le r_m^{cum} = \sum_{n=0}^{m} r_n = (1-p)\sum_{n=0}^{m} p^n = (1-p)\left(\frac{1-p^{m+1}}{1-p}\right) = 1-p^{m+1}$$

The summation identity is easily verified from the above expansion of 1/(1-*p*). Some rearrangements then give the result that the median runlength corresponds to the smallest integer *m* that satisfies the relation

$$m \ge -\left(\frac{\log(2)}{\log(p)} + 1\right)$$

It is interesting that the median runlength $\tilde{r}$ =*m* is always less than the mean runlength $\bar{r}$ =*p*/(1-*p*), as demonstrated in the following table.

Table 5.2. Runlength statistics for selected shot probabilities *p*.

p	$\bar{r}$ =*p*/(1-*p*)	-(1+log(2)/log(p))	$\tilde{r}$	$\tilde{r}$ /$\bar{r}$
0.5	1.0	0.0	0	0.000
0.6	1.5	0.4	1	0.667
0.7	2.3	0.9	1	0.429
0.8	4.0	2.1	3	0.750
0.9	9.0	5.6	6	0.667
0.91	10.1	6.3	7	0.692
0.92	11.5	7.3	8	0.696
0.93	13.3	8.6	9	0.677
0.94	15.7	10.2	11	0.702
0.95	19.0	12.5	13	0.684
0.96	24.0	15.9	16	0.667
0.97	32.3	21.8	22	0.680
0.98	49.0	33.3	34	0.694
0.99	99.0	67.9	68	0.687

Problem 5.12: What is an approximate relation between the median and the mean runlength for the r_n distribution?

Answer: Using natural logarithms, the median runlength may be written

$$\tilde{r} \approx -\left(\frac{\ln(2)}{\ln(p)} + 1\right) = -\left(\frac{\ln(2)}{\ln\left(\frac{\bar{r}}{1+\bar{r}}\right)} + 1\right)$$

For reasonably large $\bar{r}$, the denominator simplifies using the approximations

$$\frac{\bar{r}}{1+\bar{r}} = 1 - \frac{1}{\bar{r}} + \left(\frac{1}{\bar{r}}\right)^2 - \ldots \approx 1 - \frac{1}{\bar{r}}$$

$$\ln\left(\frac{\bar{r}}{1+\bar{r}}\right) \approx \ln\left(1-\frac{1}{\bar{r}}\right) = -\left(\frac{1}{\bar{r}}\right) + \tfrac{1}{2}\left(\frac{1}{\bar{r}}\right)^2 - \tfrac{1}{3}\left(\frac{1}{\bar{r}}\right)^3 + \ldots \approx -\left(\frac{1}{\bar{r}}\right)$$

This gives the approximate relation