executed correctly, the stripe will appear "stationary" as the ball rolls. A small error in the contact point, or in the ball setup, will result in a small wobble of the stripe on the rolling ball.

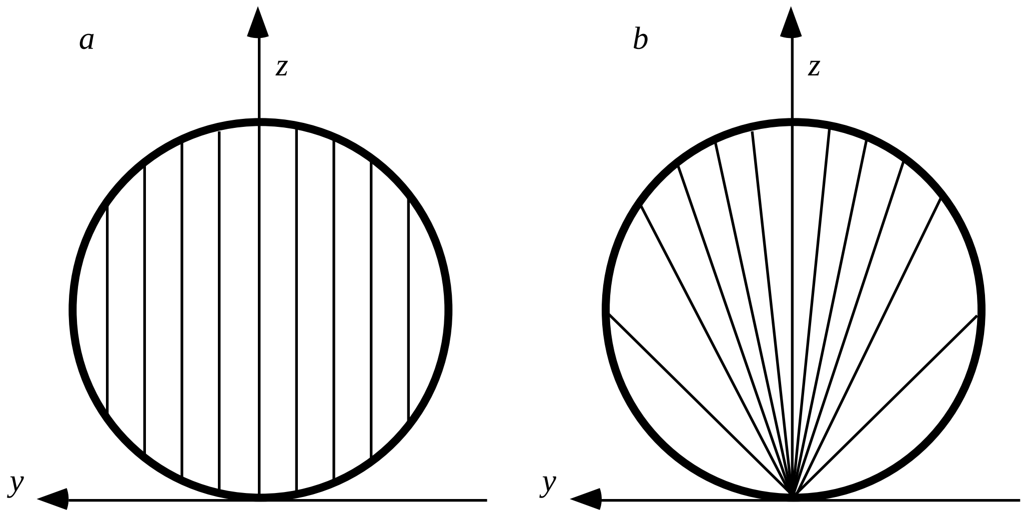

Fig. 3.1. The cue tip contact points corresponding to various arbitrary sidespin/speed ratios are denoted by the thin lines as viewed from the rear of the cue ball. Figure *a* denotes constant spin/speed ratios immediately after contact with the cue tip; these are vertical straight lines. Figure *b* denotes constant spin/speed ratios after natural roll is achieved; these are straight lines that all intersect at the point $(y,z)=(0,0)$. In both cases, the larger offsets from the center are associated with higher spin/speed ratios.

Problem 3.6: Which cue tip contact points will result in the same sidespin/speed ratios immediately after contact with the cue tip? Which contact points will result in the same sidespin/speed ratios after the cue ball achieves natural roll?

Answer: Consider the coordinate axes in Fig. 3.1. The *z*-coordinate is the height above the cloth, and the *y*-coordinate is the distance away from the vertical plane through the center of the ball. $b_y=y$ is the horizontal impact parameter, and $b_z=(z-R)$ is the vertical impact parameter. Denote the point of contact with coordinates (y,z). In terms of the linear momentum p, the initial forward velocity and forward rotation are given by

$$V_0=\frac{p}{M}$$

$$\omega_{0y}=\frac{p(z-R)}{I}=\frac{5p(z-R)}{2MR^2}\quad .$$

The forward rotation depends only on the height of the cue tip contact point z and not on the sideways displacement y. Upon achieving natural roll, the final forward velocity (see P2.2) is given by

$$V_{NR}=\tfrac{5}{7}V_0+\tfrac{2}{7}R\omega_{0y}=\frac{5p}{7M}+\frac{5p(z-R)}{7MR}=\frac{5p}{7M}\left(\frac{z}{R}\right)\quad .$$

The sidespin (*i.e.* the angular velocity about the vertical axis) is assumed to be unchanged by the frictional forces of the sliding ball. From P3.4, the initial, and final, sidespin about the z-axis is given by

$$\omega_z = \left(\frac{5y}{2R^2}\right)V_0 = \frac{5yp}{2R^2 M} \quad .$$

The sidespin depends only on the horizontal displacement, y. The sidespin/speed ratio for the initial velocity is given by

$$J_z = \frac{R\omega_z}{V_0} = \frac{5y}{2R}$$

This ratio depends only on the horizontal impact parameter y, and is independent of the ball speed V_0 and vertical contact point z. The same ratio would occur with a soft hit as with a very hard hit.

Taking the ratio of the sidespin and final natural roll velocity gives

$$J_{z,NR} = \frac{R\omega_z}{V_{NR}} = \frac{7}{2}\left(\frac{y}{z}\right)$$

where $J_{z,NR}$ is the desired spin/speed ratio. The set of points (y,z) that correspond to the same $J_{z,NR}$ are given by the straight line defined by

$$z = \left(\frac{7}{2J_{z,NR}}\right)y$$

The lines corresponding to several $J_{z,NR}$ are shown in Fig. 3.1. It is interesting that exactly the same effect may be obtained by striking the cue ball at any point on a given straight line, provided the cue ball has sufficient time to achieve natural roll through sliding friction. For a desired final velocity, a higher initial velocity is required for small-z contact points in order to overcome the drag. Note that higher sidespin/speed ratios (larger $J_{z,NR}$) are associated with straight lines closer to horizontal, and smaller ratios (smaller $J_{z,NR}$) are associated with more vertical slopes.

Problem 3.7: Of the set of points (y,z). that correspond to a constant natural-roll spin/speed ratio $J_{z,NR}$, which point (y_0,z_0) corresponds to the smallest displacement from center ball?

Answer: Consider Fig. 3.2. All the points a given distance from center ball will form a circle. The smallest circle that touches the desired straight line, as determined in P3.6, will define the smallest displacement that gives the desired spin/speed ratio. The point at which this smallest circle touches the appropriate straight line is denoted (y_0,z_0). At this point, the curve defining the circle and the straight line will be tangent, and the three points (0,0), (0,R), and (y_0,z_0) will form a right triangle. Let α be the angle away from vertical as indicated in Fig. 3.2. The tangent of this angle is given by $\tan(\alpha)=y_0/z_0$, and also by $\tan(\alpha)=(R-z_0)/y_0$. Equating these two expressions gives

$$y_0^2 = z_0(R - z_0).$$

Completing the square on the right hand side of this equation and rearranging gives

$$y_0^2 + \left(z_0 - \tfrac{1}{2}R\right)^2 = \left(\tfrac{1}{2}R\right)^2 \; .$$

This is recognized as the equation for a circle of radius $\frac{1}{2}R$ centered at the point $(0, \frac{1}{2}R)$. Contacting these points with the cue tip is called *aiming on the small circle*. When a player aims on the small circle, and the cue ball subsequently achieves natural roll, the desired spin/speed ratio $J_{z,NR}$ is achieved with the minimal displacement from center ball. It is possible to achieve much higher spin/speed ratios when the cue ball is allowed to achieve natural roll than the ratios that can be obtained immediately after cut tip contact as demonstrated in the following problem.

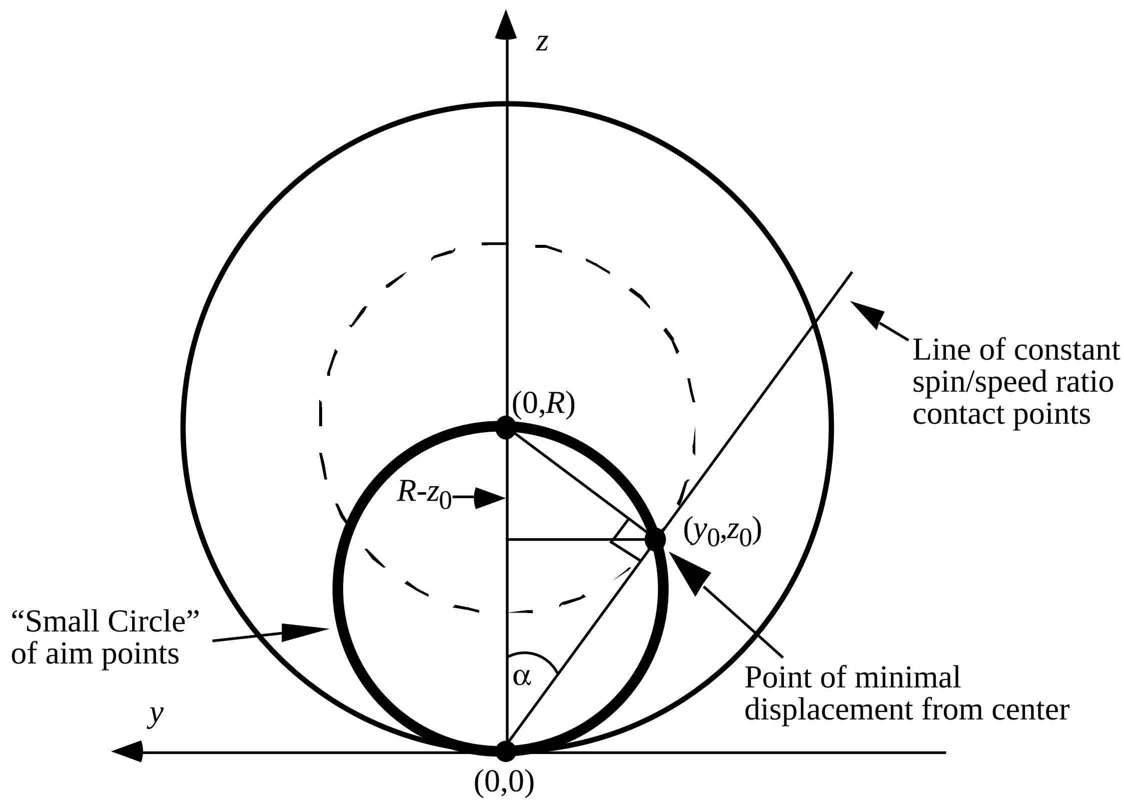

Fig. 3.2. The set of points that correspond to the minimal displacements from center ball for various spin/speed ratios after natural roll is achieved fall on a small circle of radius $R/2$ that touches the bottom point of the cue ball.

Problem 3.8: What is the natural roll sidespin/speed ratio, $R\omega_z/V_{NR}$, for the equatorial cue tip contact point $P_1=(y_1,z_1)=\left(\frac{1}{\sqrt{2}}R, R\right)$? What is the natural roll sidespin/speed ratio for the contact point $P_2=(y_2,z_2)=\left(\frac{1}{2}R, \frac{1}{2}R\right)$? At what contact points $P_3=(y_3,z_3)$ would the initial spin/speed ratio, $R\omega_z/V_o$, be the same as the natural roll spin/speed ratio of P_2?
Answer: From P3.6 the natural roll spin/speed ratio for P_1 is given by

$$\left(\frac{R\omega_z}{V_{NR}}\right) = \frac{7}{2}\left(\frac{y_1}{z_1}\right) = \frac{7}{2\sqrt{2}} = 2.475$$

The natural roll spin/speed ratio for P_2 is

$$\left(\frac{R\omega_z}{V_{NR}}\right) = \frac{7}{2}\left(\frac{y_1}{z_1}\right) = \frac{7}{2} = 3.5$$

Although the displacements away from center of these two points are the same, namely $R/\sqrt{2}$, the sidespin/speed ratio for the second point is over 41% larger than the first point. The second point P_2 is on the "small circle" and therefore results in the maximal natural roll sidespin/speed ratio for this displacement distance.

In order to achieve a comparable initial sidespin/speed ratio

$$\frac{7}{2} = \frac{R\omega_z}{V_0} = \frac{5}{2}\left(\frac{y}{R}\right)$$

$$P_3 = (y_3, z_3) = \left(\tfrac{7}{5}R, z\right)$$

However, the set of points P_3 are not on the cue ball. Therefore, it is impossible to achive such a large sidespin/speed ratio without taking advantage of the drag to reduce the ball velocity. For practical purposes, a sidespin/speed ratio of 3.5 is about as large as can be attained with a cue tip impact with a level cue stick. Larger ratios can be achieved only with elevated cue stick strokes (masse) or with collisions involving other balls.

It is sometimes convenient to think of the cue ball spin and velocity at any moment in time for a sliding ball in terms of an "effective cue tip contact point". That is, for a given linear and angular velocity of a cue ball, there exists a contact point on the cue ball at which, if the cue tip where to strike a stationary ball at that point, with the correct velocity, the result would be to match, or to reproduce, exactly the same spin and speed. Because the linear and angular velocities change as the ball slides, the effective contact point is time dependent. From P3.6, the horizontal and vertical components of the spin are related to the vertical and horizontal components of the impact parameter of the cue tip contact point according to

$$\frac{5}{2}\left(\frac{b_y^{eff}}{R}\right) = \frac{R\omega_z}{V} = \frac{R\omega_{0z}}{(V_0 - \mu g t)}$$

$$\frac{5}{2}\left(\frac{b_z^{eff}}{R}\right) = \frac{R\omega_y}{V} = \frac{\left(R\omega_{0y} + \tfrac{5}{2}\mu g t\right)}{(V_0 - \mu g t)}$$

where the time dependence of the angular and linear velocities due to the cloth friction on the sliding ball from Section 2 have been used. The origin t=0 is taken in the above equations to be the time at which the cue tip strikes the ball.

Problem 3.9: Show that the set of effective contact points corresponding to $b_y^{eff}(t)$ and $b_z^{eff}(t)$ for a sliding ball lie on a straight line passing through the coordinate points $(y,z)=(0,0)$ and $(y,z)=(b_y^{eff}(0),R+b_z^{eff}(0))$.

Answer: Let b_z^{eff} be considered as a function of b_y^{eff} and defined parametrically through the time variable t. Solve the first equation above for t in terms of b_y^{eff}, and substitute into the second to give

$$\frac{\left(R+b_z^{eff}(t)\right)}{b_y^{eff}(t)}=\frac{\left(R+b_z^{eff}(0)\right)}{b_y^{eff}(0)}$$

The right hand side of this equation is time independent. Therefore, the slope of the curve defined by the points $(y,z)=(b_y^{eff}(t),R+b_z^{eff}(t))$ is a constant, independent of time, and the set of time-dependent effective contact points lie on a straight line. The distance $(R+b_z^{eff}(t))$ is the height of the tip contact point above the cloth as seen for example in Fig. 3.2, and the distance $b_y^{eff}(t)$ is the horizontal tip displacement. Therefore, the line passing through the point (0,0) at the bottom of the ball to the initial point $(b_y^{eff}(0),R+b_z^{eff}(0))$ has the same slope as the rest of the line. The line segment of effective contact points ends when $b_z^{eff}(t)=\frac{2}{5}R$, at which time the ball achieves natural roll.

The result of P3.9 allows the player to compensate accurately for the effects of table friction on the spin axis with the following approach. First determine the desired spin axis at the eventual position of the cue ball. A stun shot for example, which is a frequent goal, would have a vertical spin axis at the time the cue ball collides with the object ball. This spin axis corresponds to some effective contact point $(b_y^{eff}(t),R+b_z^{eff}(t))$. In the case of a stun shot, this point would have coordinates $(b_y^{eff}(t),R)$ and correspond to pure sidespin. The player must then estimate, based on shot speed and the cloth friction, the required vertical offset below center in order to achieve a stun shot. Let this vertical distance be denoted δ. The player then draws an imaginary line from the point $(b_y^{eff}(t),R)$, corresponding to the desired target spin state of the cue ball, to the point (0,0). The point on that imaginary line that corresponds to $(b_y^{eff}(0),R-\delta)$ is the desired contact point. Other final spin states would be estimated in the same manner. The straight line is always drawn from the final effective contact point to the origin (0,0), and the player works backward in time, so to speak, from the final spin state of the cue ball to the initial tip/ball contact time. If, during this process, the actual contact point $(b_y^{eff}(0),b_z^{eff}(0))$ is judged to be outside the boundary at which miscues begin to occur (see P1.7), then the desired shot is not possible, and the player should seek other alternatives.

Problem 3.10: What is the relation between the cue stick velocity immediately before contact, the cue ball velocity immediately after contact, and the impact parameter b? (assume that the total kinetic energy is conserved)

Answer: Conservation of linear momentum and kinetic energy give

$$M_s V_0 = M_s V_s + M_b V_b$$

$$\tfrac{1}{2} M_s V_0^2 = \tfrac{1}{2} M_s V_s^2 + \tfrac{1}{2} M_b V_b^2 + \tfrac{1}{2} I \omega_b^2$$

$$= \tfrac{1}{2} M_s V_s^2 + \left(\tfrac{1}{2} + \tfrac{5}{4} \left(\frac{b}{R} \right)^2 \right) M_b V_b^2$$

Solve the first equation for V_s, and substitute into the second equation to obtain

$$V_b = \frac{2V_0}{1 + \dfrac{M_b}{M_s} + \tfrac{5}{2} \left(\dfrac{b}{R} \right)^2}$$

It may be verified that this expression agrees with that of P3.2 when b=0. It may now be understood why it is desirable to avoid spin on the cue ball during the break shot. For a given cue stick energy, or velocity V_0, any spin corresponding to nonzero b has the effect of reducing the cue ball velocity and the translational kinetic energy; the maximum cue ball speed is achieved with a centerball b=0 contact point. The ratio V_b/V_0 is plotted as a function of impact parameter for some selected ball/stick mass ratios in Fig. 3.3.

Problem 3.11: What is the vertical impact parameter that maximizes the ratio V_{NR}/V_0 where V_{NR} is the cue ball natural roll velocity and V_0 is the before-collision cue stick velocity?

Answer: From P2.2, P3.6, and P3.10, the natural roll velocity is given by

$$V_{NR} = \tfrac{5}{7} V_b + \tfrac{2}{7} R \omega_b = \left(\tfrac{10}{7}\right) \left(\frac{1 + \left(\dfrac{b}{R} \right)}{1 + \dfrac{M_b}{M_s} + \tfrac{5}{2} \left(\dfrac{b}{R} \right)^2} \right) V_0$$

Solving for the velocity ratio, differentiating with respect to b, setting the result to zero, and simplifying gives

$$\left(\frac{b}{R} \right)_{\max V_{NR}} = -1 + \sqrt{\frac{7}{5} + \frac{2}{5} \left(\frac{M_b}{M_s} \right)}$$

For a 6*oz* ball and an 18*oz* stick, the optimal impact point is given by

$$\left(\frac{b}{R} \right)_{\max V_{NR}} = 0.238 \qquad [M_s/M_b=3]$$

and for a 24 *oz* stick the optimal impact point is

$$\left(\frac{b}{R} \right)_{\max V_{NR}} = 0.225 \qquad [M_s/M_b=4]$$

This range includes most common stick weights and shows that the optimal impact point is only weakly dependent on the stick weight in this range. In both cases, the impact point is between centerball b=0 and the natural roll height $b=\tfrac{2}{5}R$. The initial cue ball velocity is maximized at b=0, but $\tfrac{2}{7}$ of this velocity is lost upon achieving natural roll to sliding friction; at $b = \frac{2}{5}R$ there is no velocity loss due to sliding friction, but the initial velocity is relatively small due to the energy and momentum transfer conditions between the stick and ball. The above contact point is the optimal compromise between these two extremes. Maximization of the natural roll velocity is the same as maximizing the natural roll energy, and is the same as maximizing the distance that the ball rolls before stopping due to rolling resistance. Because this distance is maximized, this also means that the distance is relatively insensitive to small deviations of the contact point away from this optimal value. This is most useful when cue ball placement is of utmost importance such as, for example, during the lag shot at the beginning of a match. The ratio V_{NR}/V_0 is plotted as a function of impact parameter for some selected ball/stick mass ratios in Fig. 3.4.

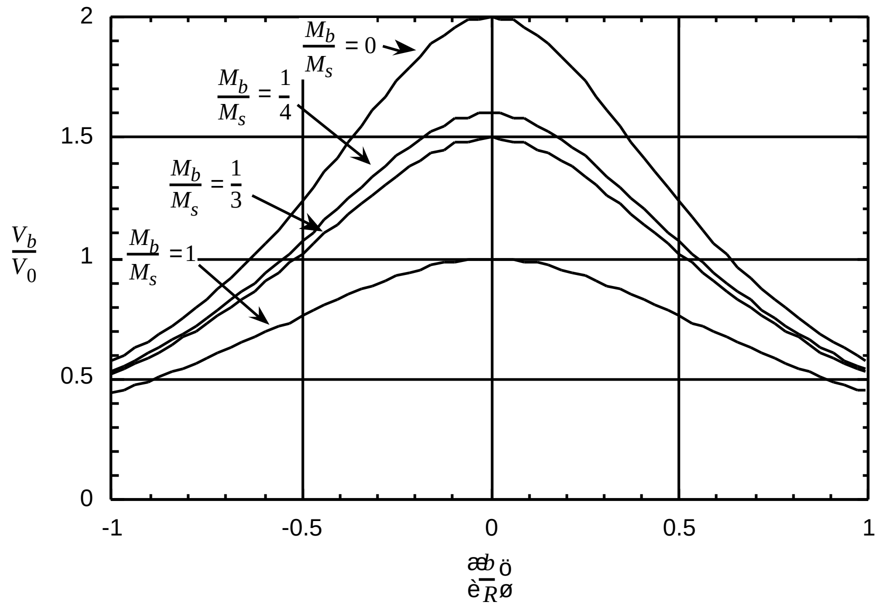

Fig 3.3. The ratio of the cue ball velocity V_b to the before-collision cue stick velocity V_0 is shown as a function of the vertical impact parameter (b/R) for some selected ball/stick mass ratios.

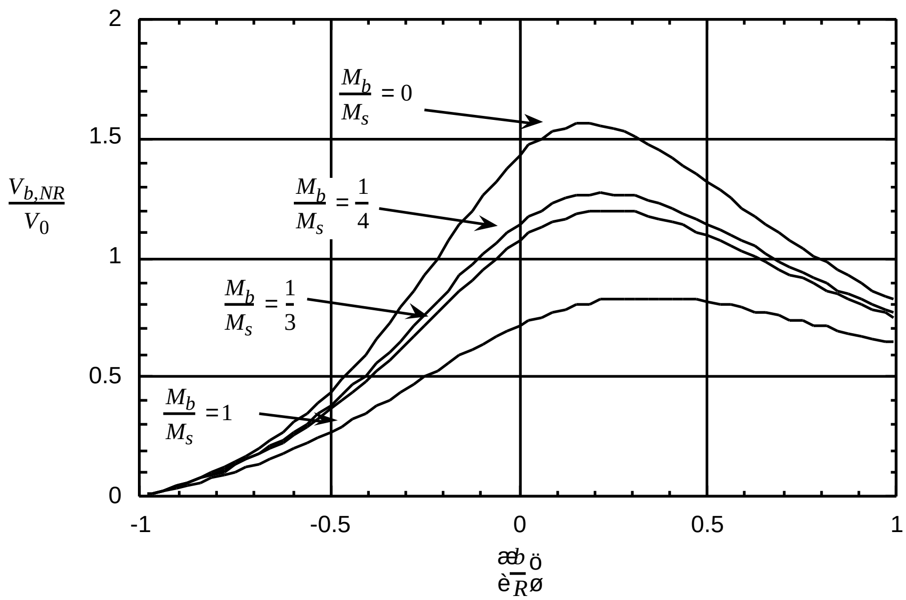

Fig. 3.4. The ratio of the final natural roll cue ball velocity $V_{b,NR}$ to the before-collision cue stick velocity V_0 is shown as a function of the vertical impact parameter (b/R) for some selected ball/stick mass ratios. For a given ball/stick mass ratio, the optimal contact point for a lag shot is determined by the flat region near the curve maximum.

4. Collisions Between Balls

Consider the motions of two colliding balls. One ball is assumed to be moving before the collision, and both balls are assumed to be moving afterwards. For this discussion, assume that the initially moving ball is the cue ball, and the initially stationary ball is an object ball. As the two balls collide in an off-center hit, the frictional forces acting tangential to the surfaces are relatively small (e.g. compared to the frictional forces between a ball and the cue tip). All of the remaining force is directed along the line between the centers of the balls.

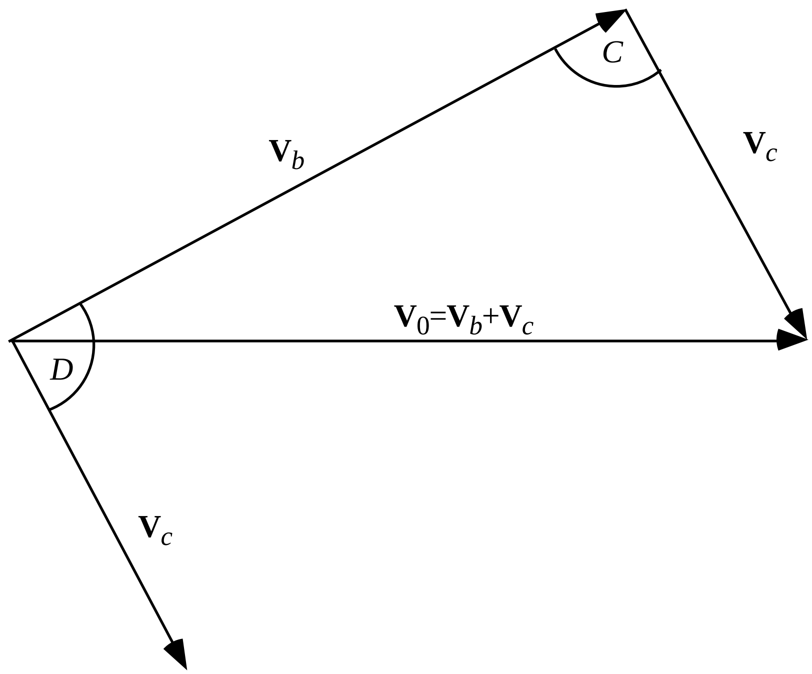

Fig. 4.1. Pictorial representation of the conservation of momentum vector relation $\mathbf{V}_0=\mathbf{V}_b+\mathbf{V}_c$. The angles C and D are supplementary and satisfy the relation $C+D=\pi$.

Consider first the ball motions just before the collision and just after the collision; in this situation, the friction between the cloth and the sliding/rolling balls has not had time to affect the ball trajectories. Linear momentum ($\mathbf{p}=M\mathbf{V}$) is conserved in both the *x*- and *y*-coordinate directions. Represented with vectors, the vector sum of the final momentum of the two balls is equal to the initial momentum of the cue ball. Eliminating the mass *M* of the balls, results in the vector relation $\mathbf{V}_0=\mathbf{V}_b+\mathbf{V}_c$ between the initial and final velocities. This relation is shown pictorially in Fig. 4.1. The final velocity of the cue ball $\mathbf{V}_c$ has been drawn twice: once with its base common to that of the $\mathbf{V}_b$ vector, which is consistent with both balls departing from the same collision point on the table, and again with its base at the end of the $\mathbf{V}_b$ vector to show pictorially that $\mathbf{V}_0=\mathbf{V}_b+\mathbf{V}_c$. The angles *D* and *C* are supplementary and are related by (in radians) $C+D=\pi$, and consequently, $\cos(C)=-\cos(D)$.

In addition to momentum, energy is also conserved in this collision to a good approximation. The relatively small amount of energy that is lost is turned into sound or heat within the balls. An *elastic* collision is one in which energy is assumed to be conserved, so this energy loss will be denoted $E_{inelastic}$. As discussed in the previous sections, there are two kinds of kinetic energy, translational and rotational, associated with each ball. Equating the energy before and after the collision gives

$$T_{0(Trans)} + T_{0(Rot)} = T_{c(Trans)} + T_{c(Rot)} + T_{b(Trans)} + T_{b(Rot)} + E_{inelastic}$$

Collecting all the $T_{(Rot)}$ terms together, and multiplying by 2/M gives the relation

$$V_0^2 = V_b^2 + V_c^2 + \Delta_{elastic} + \Delta_{inelastic} = V_b^2 + V_c^2 + \Delta_{total}$$

with

$$\Delta_{elastic} = \frac{2}{M}\left(T_{c(Rot)} + T_{b(Rot)} - T_{0(Rot)}\right)$$

$$\Delta_{inelastic} = \frac{2}{M} E_{inelastic}$$

The term $\Delta_{elastic}$ depends on the total change of rotational energy. The contribution $\Delta_{elastic}$ may be positive, zero, or negative, but the term $\Delta_{inelastic}$ is always positive, since it represents an energy loss in the collision process. There are two types of contributions to $E_{inelastic}$, the first type of energy loss is due to the frictional forces of the sliding balls. These frictional forces result in the exchange of energy between the various translational and rotational components. Just as in the case of the simple sliding block, the frictional forces are intimately related to the inelastic energy loss; without this inelastic energy loss, there would be no sliding friction. As will be seen in the following discussions, this inelastic energy loss can be determined by analysis of the resulting momentum exchange between the balls. Other contributions to the inelastic energy loss involve the imperfect transfer of energy between the balls. For example, the sound made by the colliding balls represents a transfer of kinetic energy from the collision process to the surroundings. This energy loss would occur even in the absence of sliding frictional forces. In the present discussion, this latter type of energy loss will not be considered quantitatively in the analysis. With this simplification, both the elastic and inelastic contributions to Δ_{total} are assumed to be associated with the tangential forces of sliding friction.

The law of cosines for an arbitrary triangle with sides *a, b,* and *c* with corresponding angles *A*, *B*, and *C* is

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

This allows the angles of a triangle to be related to the lengths of the three sides. In particular, the sides of the triangle resulting from the pictorial representation of the conservation of momentum relation may be related to the departure angle. Comparing the law of cosines with the above velocity equation gives the relation

$$\cos(C) = \frac{-\Delta_{total}}{2|V_b||V_c|} = -\cos(D)$$

between the angles C and D and the change of translational energy term Δ_{total}. If there is no translational kinetic energy loss during the collision of the balls, then $\Delta_{total}=0$, $\cos(C)=0$, and $C=\pi/2$ is a right angle (*i.e.* 90 degrees). In this case, the law of cosines reduces to the familiar theorem of Pythagoras. If $C=\pi/2$, then $D=\pi/2$ and the two balls depart at exactly a right angle. In this initial discussion it will be assumed that the balls are rotating about the vertical axes only; the more general situation is examined later. If there is no rotational energy change during the collision, then $\Delta_{elastic}=0$. There are three situations in which there will be no total rotational energy change during a collision. First, if there is no friction between the balls, then there will be no tangential forces acting at the point of contact. This is, of course, an approximation, but for many shots such an approximation is sufficient, and in any case it defines a convenient reference point. The second situation in which no spin change occurs is when the cue ball has just the right amount of outside spin so that the ball surfaces are not moving relative to each other during the (very short) collision time. In this case the cue ball spin is unchanged, and the object ball acquires no spin during the collision. The third situation in which no total rotational energy change occurs is when the cue ball has just the right amount of inside spin so that all of the cue ball spin is transferred to the object ball, and the cue ball departs with no spin. The first situation is an ideal, and occurs only with no friction between the colliding balls; $\Delta_{total}=0$ in this case for all collision situations. The second situation is independent of the ball friction, but depends on matching exactly the outside spin and the cut angle; $\Delta_{total}=0$ for this situation since both components vanish when there is no friction. The third situation depends on matching the amount of inside spin with the friction between the balls and the cut angle; since there are accelerations associated with the frictional forces, there is a nonzero $\Delta_{inelastic}$ component, $\Delta_{total}\neq 0$, and therefore the departure angle will differ from $\pi/2$.

To appreciate the importance of spin transfer, consider a cut shot, with ball friction, when the cue ball has no spin initially. In this case, the $T_{0(Rot)}$ term will be zero, but both $T_{c(Rot)}$ and $T_{b(Rot)}$ will be nonzero. The cue ball acquires some sidespin by rubbing against the object ball, and the initially motionless object ball acquires some sidespin by rubbing against the cue ball. In this case, both $\Delta_{elastic}>0$ and $\Delta_{inelastic}>0$, the angle C will be larger than $\pi/2$, and the angle of departure D will be smaller than a right angle. In actual practice this is a small effect, in the neighborhood of 2-4 degrees depending on how sticky are the pair of colliding balls, but a 4 degree angle, over 8 feet results in a deviation of 6.7", or about half a diamond on a 9' table ($\tan(\alpha)=d/L$ with deviation angle α, distance L, and deviation distance d). When referring to the resulting object ball deviations, this effect is called *collision-induced throw*, and clearly this must be accounted for, to some extent, on any but the most trivial of shots.

Problem 4.1: What are the conditions in which $\Delta_{elastic}$ will be positive, zero, and negative? (assume all spins are about the vertical axes)

Answer: Substituting the rotational energy expression gives

$$\Delta_{elastic} = \tfrac{2}{5} R^2 \left(\omega_c^2 + \omega_b^2 - \omega_0^2 \right)$$

where all angular velocities are relative to the vertical axes of each ball. However, any change of angular velocity in the cue ball must be compensated exactly by a corresponding change in the object ball angular velocity, since the frictional forces on each ball are equal but opposite in direction.

$$\omega_0 = \omega_c - \omega_b \ .$$

Substitution of this relation gives

$$\Delta_{elastic} = \tfrac{4}{5} R^2 \omega_b \omega_c \ .$$

When the final spins of both balls are in the same direction (*i.e.* both are clockwise when looking down on the table from above, or both are counterclockwise), then $\Delta_{elastic}$ will be positive, cos(*D*) will be positive, and the angle of departure of the two balls will be $<\pi/2$. When the final spin of either the cue ball or the object ball is zero, then $\Delta_{elastic}$ will be zero, and the departure angle will be $\leq\pi/2$, and the magnitude will depend entirely on $\Delta_{inelastic}$ which is always nonnegative. These are the only situations that result in $\Delta_{elastic}=0$. When the final spins of the two balls are in opposite directions (*i.e.* one clockwise and the other counterclockwise), then $\Delta_{elastic}$ will be negative, and the departure angle will depend on the relative magnitudes of the two components $\Delta_{elastic}$ and $\Delta_{inelastic}$. Note that cos(*D*) depends on the final spin/speed ratios of the balls, so within the current set of simplifying approximations, the contribution of $\Delta_{elastic}$ to the departure angle is independent of the overall shot speed.

The above qualitative analysis did not require a detailed examination of the forces during the collision process. These forces and the resulting ball trajectories are now examined in more detail. For this pupose, it is useful to define two coordinate systems as shown in Fig. 4.2. The first coordinate system, denoted (*x*',*y*',*z*'), is appropriate for the initial cue ball velocity before the collision; the second, denoted (*x*,*y*,*z*) is the natural coordinate system to describe the trajectories after the collision. Unit vectors along these two coordinate axes satisfy the transformation relation

$$\begin{pmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix}$$

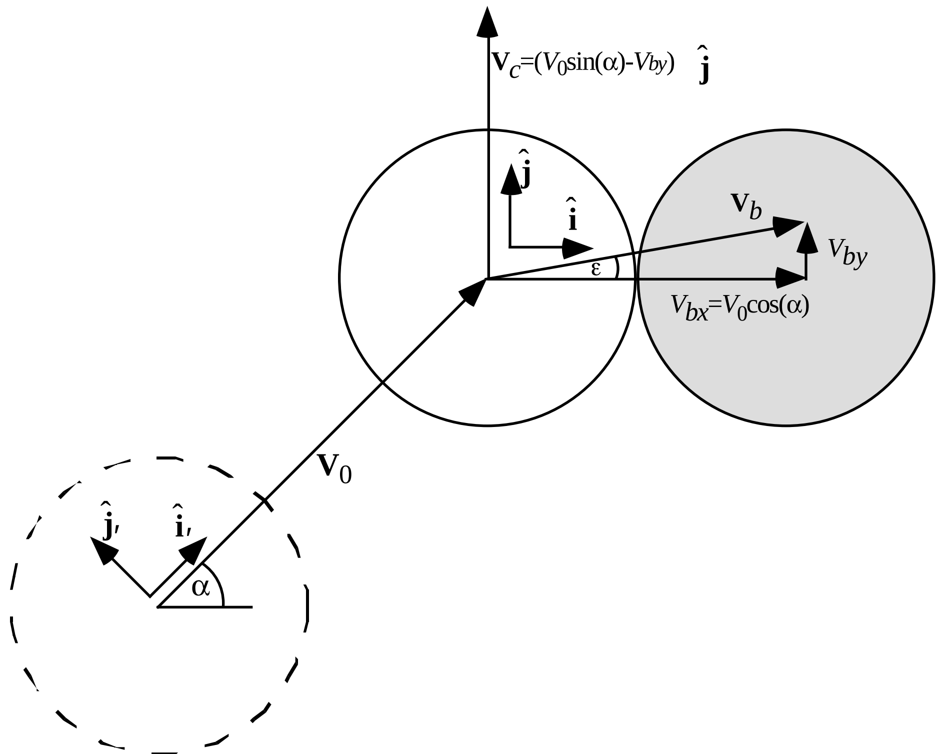

Fig. 4.2. The effects of the sliding frictional forces on the object ball and cue ball are shown in detail on the after-collision velocity vectors. Two coordinate systems are used in the analysis of object ball throw. The first is relative to the initial cue ball velocity $\mathbf{V}_0$, the second is appropriate to describe the after-collision velocities. The vertical *z*-coordinate is not shown, but is directed out of the plane of the figure. The angle α would be the object ball cut angle if there were no friction.

It is convenient to take the origin of the (*x*,*y*,*z*) coordinate system to be the cue ball center at the moment of contact with the object ball. With this choice, the contact point of the cue ball and object ball lies on the *x*-axis. In the absense of friction, the object ball would depart along the *x*-axis and the cue ball would depart along the *y*-axis. The frictional forces are tangential to the point of contact, and therefore lie in the *yz* plane. The direction of the frictional force is determined by the velocity of the contact point of the cue ball at the moment of contact. The contact point velocity is the sum of the linear velocity

$$\mathbf{V}_0 = V_0\hat{\mathbf{i}}' = V_0\left(\cos(\alpha)\hat{\mathbf{i}} + \sin(\alpha)\hat{\mathbf{j}}\right)$$

and the angular velocity

$$\omega_0 \times \mathbf{r}_{cp} = R\,\omega_0 \times \hat{\mathbf{i}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_{0x} & \omega_{0y} & \omega_{0z} \\ R & 0 & 0 \end{vmatrix} = R\left(\omega_{0z}\hat{\mathbf{j}} - \omega_{0y}\hat{\mathbf{k}}\right)$$

If the cue ball is struck with a level cue stick (*i.e.* no masse), then the cue ball rotation may be written as

$$\omega_0 = \omega'_{0y}\hat{\mathbf{j}}' + \omega'_{0z}\hat{\mathbf{k}}' = -\omega'_{0y}\sin(\alpha)\hat{\mathbf{i}} + \omega'_{0y}\cos(\alpha)\hat{\mathbf{j}} + \omega_{0z}\hat{\mathbf{k}}$$

ω'_{0y}<0 for backspin, ω'_{0y}=0 for a stun shot, and ω'_{0y}>0 for topspin. $\omega'_{0z} = \omega_{0z}$ corresponds to sidespin. The resulting contact point velocity is

$$\begin{aligned}\mathbf{V}_{cp} &= V_0\cos(\alpha)\hat{\mathbf{i}} + \left(V_0\sin(\alpha) + R\omega_{0z}\right)\hat{\mathbf{j}} - R\omega'_{0y}\cos(\alpha)\hat{\mathbf{k}} \\ &= V_{cpx}\hat{\mathbf{i}} + V_{cpy}\hat{\mathbf{j}} + V_{cpz}\hat{\mathbf{k}}\end{aligned}$$

It is the sign of V_{cpy} that determines the direction of throw of the object ball. V_{cpy} >0 results in throwing the object ball in the $+\hat{\mathbf{j}}$ direction, V_{cpy} <0 results in $-\hat{\mathbf{j}}$ throw, and V_{cpy} =0 results in no throw. It is interesting that, for a given angle α, V_{cpy} depends only on the cue ball sidespin ω_{0z}. Cue ball topspin or draw does not change the direction of throw, but it does change the magnitude of the throw.

The V_{cpx} component of the contact point velocity is directed exactly along the object ball center of mass. As the balls collide, the momentum component $p_x = MV_{cpx}$ is transferred entirely from the cue ball to the object ball. This momentum is transferred during the very short collision time *t* according to the equation $p_{bx} = \int_0^t F_x(t')dt'$. If there are any tangential components of the contact point velocity, then at any time during the collision there is a tangential frictional force with magnitude given by $F(t) = \mu_{bb}F_x(t)$ where μ_{bb} is the ball-ball sliding coefficient of friction. The direction of this tangential force is determined by the tangential components of the contact point velocity. A unit vector in this tangential direction may be defined as

$$\begin{aligned}\hat{\mathbf{e}} &= \frac{\mathbf{V}_{cp}}{|\mathbf{V}_{cp}|} = \frac{V_{cpy}\hat{\mathbf{j}} - V_{cpz}\hat{\mathbf{k}}}{|V_{cpy}\hat{\mathbf{j}} - V_{cpz}\hat{\mathbf{k}}|} \\ &= \frac{\left(V_0\sin(\alpha) + R\omega_{0z}\right)\hat{\mathbf{j}} - R\omega'_{0y}\cos(\alpha)\hat{\mathbf{k}}}{\left[\left(V_0\sin(\alpha) + R\omega_{0z}\right)^2 + \left(R\omega'_{0y}\cos(\alpha)\right)^2\right]^{1/2}} \\ &= \cos(\gamma)\hat{\mathbf{j}} + \sin(\gamma)\hat{\mathbf{k}}\end{aligned}$$

with obvious definitions for the horizontal component cos(γ) and vertical component sin(γ). The vertical component of this force direction sin(γ) either works in conjuction or opposition to the weight of the ball; it does not affect the *direction* of the cue ball or object ball velocities in the plane of the table after the collision. However, the horizontal component of the force cos(γ) does affect the object ball direction. It is this horizontal component of the force that results in the object ball throw. Fig. 4.3 shows the possible combinations of directions for the unit vector $\hat{\mathbf{e}}$ and the geometrical meaning of the components cos(γ) and sin(γ). The factor cos(γ) may be thought of as a *geometrical efficiency factor* in converting the frictional forces into throw velocities.

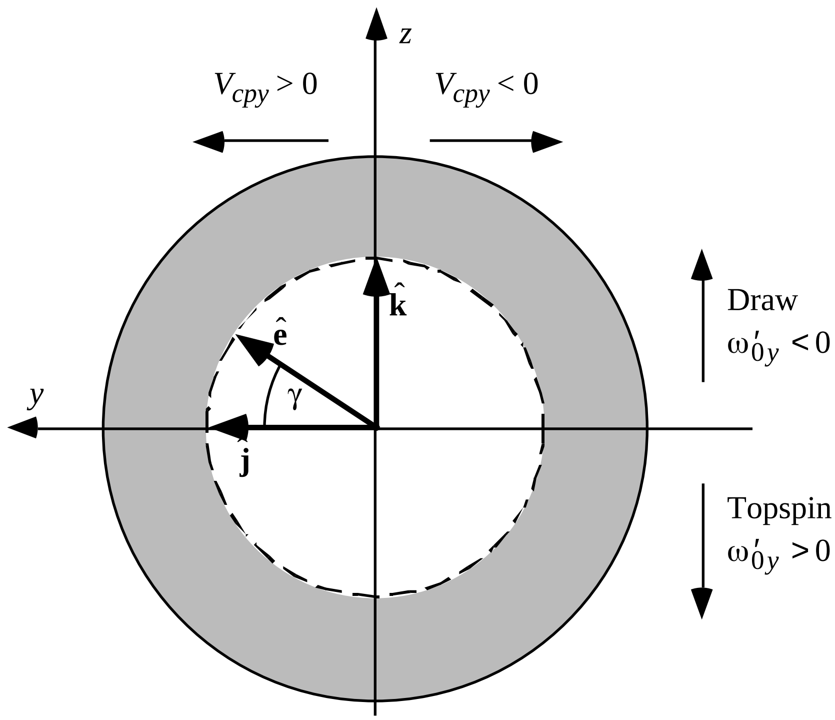

Fig. 4.3. The unit vector $\hat{\mathbf{e}}$, parallel to the direction of the sliding frictional force on the object ball, is decomposed into the horizontal and vertical components characterized by the angle γ. This force is applied to the object ball at the contact point, and an opposing force is applied to the cue ball. This force is tangential to the ball surfaces and lies in the *yz*-plane. The direction of the unit vector depends on the cut angle and the spin axis of the cue ball at the moment of the collision. The object ball throw is proportional to the horizontal component of the frictional force.

The object ball throw is determined by the *y*-component of the frictional force. Substitution of the above decomposition of $\hat{\mathbf{e}}$ gives the relations

$$p_{by} = \int_0^t F_y(t')dt' = \cos(\gamma)\mu_{bb}\int_0^t F_x(t')dt' = \cos(\gamma)\mu_{bb}p_{bx}$$

$$V_{by} = \cos(\gamma)\mu_{bb}V_{bx}$$

The horizontal component of the tangential frictional force results in the throw velocity V_{by} being added to the object ball velocity, and the opposing frictional force acts to subtract exactly this velocity from the post-collision cue ball velocity. Because the factor cos(γ) depends on several parameters, it is useful to consider some special cases.

Problem 4.2: How does the throw angle ε defined by $\tan(\varepsilon)=V_{by}/V_{bx}$, depend on overall shot speed?

Answer: Rewriting the cos(γ) expression in terms of spin/speed ratios gives

$$\cos(\gamma) = \frac{\left(\sin(\alpha) + \frac{R\omega_{0z}}{V_0}\right)}{\left(\left(\sin(\alpha) + \frac{R\omega_{0z}}{V_0}\right)^2 + \left(\frac{R\omega'_{0y}}{V_0}\cos(\alpha)\right)^2\right)^{1/2}}$$

The geometrical factor cos(γ) is seen to depend entirely on spin/speed ratios, not overall shot speed. The throw angle is $\varepsilon=\arctan(V_{by}/V_{bx})=\arctan(\mu_{bb}\cos(\gamma))$. The velocity ratio, and therefore the throw angle ε is independent of the shot speed. In practice, this result is not entirely true; the throw angle decreases slightly for very hard shots. This change of throw angle with shot speed is due to a slight speed-dependence of μ_{bb}. Fig. 4.4 shows the dependence of the object ball throw factor cos(γ) as a function of the sidespin/speed ratio ($R\omega_{0z}/V_0$) for a specific cut angle of π/6 (a half-ball cut) for several values of the topspin/speed ratio.

Problem 4.3: For a stun shot, $\omega'_{0y}=0$, how does the throw velocity depend on the cue ball cut angle α?

Answer: For a stun shot, the cos(γ) factor reduces to the form

$$\cos(\gamma) = \frac{V_{cpy}}{|V_{cpy}|} = \frac{\left(V_0\sin(\alpha) + R\omega_{0z}\right)}{|V_0\sin(\alpha) + R\omega_{0z}|} = \pm 1 \qquad [\text{for } \omega'_{0y}=0]$$

The sign of the cos(γ) factor is determined by the initial velocity component, the cut angle α, and the sidespin ω_{0z}. The throw velocity is then given by

$$V_{by} = \pm\mu_{bb} V_{bx}$$

If the cue ball has no sidespin, then cos(γ)=+1, and $V_{by} = \mu_{bb}V_{bx}$ for the shot angle in Fig. 4.2. This result was *assumed* in P1.6, as a way to determine μ_{bb}, but it is now seen with a careful analysis that this assumption was indeed correct [provided the frozen object ball acts the same as a stun-shot collision]. The only dependence of the throw velocity on the cut angle is in the *direction* of the frictional force. Fig. 4.4 shows the dependence of the object ball throw factor cos(γ) as a function of the sidespin/speed ratio ($R\omega_{0z}/V_0$) for a stun shot. There is an abrupt change in value as V_{cpy} changes sign.

Problem 4.4: For a natural roll cue ball, $R\omega'_{0y}=V_0$ (or a reverse natural roll cue ball, $R\omega'_{0y}=-V_0$) how does the throw angle depend on the cue ball cut angle α?

Answer: For a natural roll cue ball, the cos(γ) factor reduces to the form

$$\cos(\gamma) = \frac{\left(\sin(\alpha) + \frac{R\omega_{0z}}{V_0}\right)}{\left(\left(\sin(\alpha) + \frac{R\omega_{0z}}{V_0}\right)^2 + \cos(\alpha)^2\right)^{1/2}} \qquad [\text{NR or RNR}]$$

In Fig. 4.4, this factor is plotted as a function of sidespin/speed ratio for a specific cut angle $\alpha=\pi/6$. The throw angle is determined by $\varepsilon=\arctan(\mu_{bb}\cos(\gamma))$. Although the slope is steepest in the region near $V_{cpy}=0$, the slope is not as steep in this region as that for smaller values of $|R\omega'_{0y}/V_0|$.

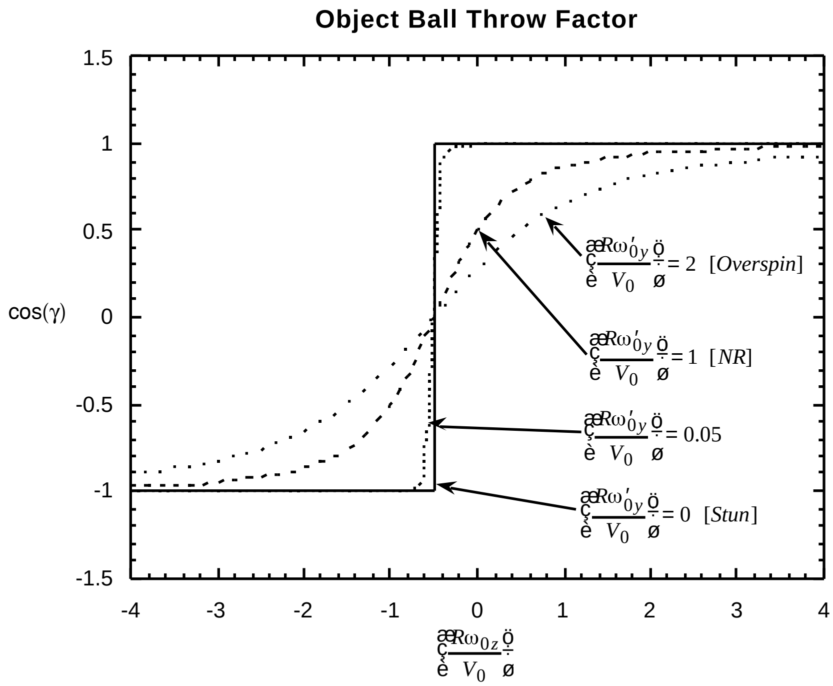

Fig. 4.4. The object ball throw factor cos(γ) is shown as a function of the cue ball sidespin to speed ratio ($R\omega_{0z}/V_0$) for selected values of of cue ball topspin/draw. The slope of the given curve determines how sensitive is the object ball throw to small variations in the sidespin.

In practice, it is impossible to achieve an exact stun shot. There will always be some small value of ω'_{0y}. Similarly, the quantity $V_{cpy}=(V_0\sin(\alpha)+R\omega_{0z})$ will never be exactly zero; it may be very small, but it will never be exactly zero. This leads to the question of how the throw angle depends on small variations from these limiting conditions. The answer is that the direction of the unit vector $\hat{\mathbf{e}}$ becomes very sensitive, rotating wildly even with very small changes in the cue ball spin. Both the numerator and the denominator of the components become small, but without a definite limit. Therefore, the cos(γ) factor can vary between –1 and +1, and the throw velocity can vary anywhere between $-\mu_{bb}V_{bx}$. and $+\mu_{bb}V_{bx}$. For small values of ω'_{0y}, the slope of the cos(γ) curve becomes very steep; this steepness reflects the sensitivity of the object ball throw to the sidespin. This correlation of steepness of slope with small ω'_{0y} values may be seen in Fig. 4.4. This slope reflects the sensitivity of the throw factor cos(γ) with respect to changes in the sidespin. The sensitivity of the throw factor with respect to changes in the topspin is related to the derivative of cos(γ) with respect to the other spin/speed ratio ($R\omega'_{0y}/V_0$).

Problem 4.5: What is the sensitivity of the object ball throw with respect to both components of the cue ball spin?

Answer: It is conventient to characterize the sensitivity in terms of the spin/speed ratios $J_{0z}=(R\omega_{0z}/V_0)$ and $J_{0y}=(R\omega'_{0y}/V_0)$. The sensitivity of the throw factor to the cue ball spin is characterized by the derivatives

$$\frac{d\cos(\gamma)}{dJ_{0y}} = \frac{-\left(\sin(\alpha)+J_{0z}\right)J_{0y}\cos^2(\alpha)}{\left(\left(\sin(\alpha)+J_{0z}\right)^2+\left(J_{0y}\cos(\alpha)\right)^2\right)^{3/2}}$$

$$\frac{d\cos(\gamma)}{dJ_{0z}} = \frac{\left(J_{0y}\cos(\alpha)\right)^2}{\left(\left(\sin(\alpha)+J_{0z}\right)^2+\left(J_{0y}\cos(\alpha)\right)^2\right)^{3/2}}$$

The first equation gives the sensitivity of the throw with respect to changes in the topspin or backspin of the cue ball, the second equation gives the sensitivity with respect to changes in the sidespin. When J_{0y} is small, then the slope of the cos(γ) factor is approximately

$$\frac{d\cos(\gamma)}{dJ_{0z}} \approx \frac{\left(J_{0y}\cos(\alpha)\right)^2}{\left|\sin(\alpha)+J_{0z}\right|^3} \qquad \text{[for small } J_{0y}\text{]}$$

This shows why the slope of the cos(γ) curve becomes essentially vertical in Fig. 4.4 as the sidespin J_{0z} passes through the zero point of V_{cpy} and the denominator of this component of the sensitivity vanishes.

A combined measure of the sensitivity of the object ball throw to the cue ball spin may be defined as

$$F(\mathbf{J}_0) = \sqrt{\left(\frac{d\cos(\gamma)}{dJ_{0y}}\right)^2 + \left(\frac{d\cos(\gamma)}{dJ_{0z}}\right)^2}$$

For values of $\mathbf{J}_0$ that correspond to small $F(\mathbf{J}_0)$, the player is allowed larger margins of error in shot execution (e.g. in the accuracy of the cue tip contact point) and in judgement (e.g. in estimating, and compensating for, the object ball throw). Regions with large $F(\mathbf{J}_0)$ are those where very small spin variations result in large changes in the object ball throw; these are the regions that the player should try to avoid. Fig. 4.5 shows a contour plot of the sensitivity F as a function of the two components of the cue ball spin, J_{0z} and J_{0y}, for the same cut angle as was used in Fig. 4.4, namely $\alpha=\pi/6$ (a half-ball cut). It may be observed that the regions of least sensitivity are those with small J_{0y} (*i.e.* close to being a stun shot), and large sidespin $|J_{0z}|$ (*i.e.* corresponding to extreme underspin or overspin). Regions of high sensitivity are seen to correspond to V_{cpy}=0 (*i.e.* to J_{0z}=-sin(α)=-½). The highest sensitivity contours correspond to the region near the point V_{cpy}=0 and J_{0y}=0; a magnified view of this region is shown in the inset in Fig. 4.5. The sensitivity of the throw angle becomes enormous in this region. Ironically, the spin combinations that result in the smallest object ball throw sometimes correspond also to the largest sensitivity, and the spin combinations that result in the largest throw sometimes correspond also to the smallest sensitivity.

With this sensitivity in mind, it is possibly a wise tactic to avoid these conditions so as to avoid the large uncertainty in the throw angle. That is, stun shots with outside spin should be avoided, according to this argument, when the effects of throw might be critical to the success of the shot. This uncertainty may be avoided in practice by ensuring that the numerator or the denominator (or both) are significantly different from zero at the moment of collision of the cue ball with the object ball. This may be done for a given shot either by avoiding stun-shot spin (*i.e.* ensuring $\omega'_{0y} \neq 0$ thereby reducing the magnitude of the cos(γ) factor), or by avoiding the V_{cpy}=0 condition (thereby producing a predictable, although nonzero throw), or by avoiding both simultaneously.

It should be pointed out that this recommendation is somewhat contrary to that given by some other players, teachers, and authors. Their argument is that minimizing the V_{cpy} factor will minimize the throw. As seen in Fig. 4.5, this is only true if $|\omega'_{0y}|$ differs from zero and is large compared to $|V_{cpy}|$. In practice for some types of shots, it may be easier to avoid the V_{cpy}=0 combinations of speed and sidespin by intentionally overspinning or underspinning the cue ball, and to account explicitly for the throw by adjusting the aim point. This approach might be preferable in situations where stun-shot spin is necessary for position. Examples of this compensation are described in the following problems. Another complicating factor is the seemingly random phenomenon called *skid* (also called *cling* or *kick*). Skid occurs when a small piece of chalk or dust is trapped between the contact point of the balls, increasing dramatically the coefficient of friction for that particular shot. When this occurs, the amount of throw associated with nonzero V_{cpy} is very unpredictable.

Problem 4.6: For a natural roll cue ball (or reverse natural roll cue ball) with no sidespin, ω_{0z}=0, how does the throw angle depend on the cue ball cut angle α?
Answer: From P4.4, the cos(γ) factor reduces to the form

$$\cos(\gamma) = \frac{(\sin(\alpha))}{\left(\sin(\alpha)^2 + \cos(\alpha)^2\right)^{1/2}} = \sin(\alpha) \qquad \text{[NR or RNR with } \omega_{0z}=0]$$

The throw angle is determined by $\varepsilon=\arctan(\mu_{bb}\cos(\gamma))=\arctan(\mu_{bb}\sin(\alpha))$. The throw depends only on the cut angle α. It is 0 for a straight in shot (α=0), and increases to a maximum value for very thin cuts (α≈π/2). The impact parameter for the cue ball/object ball collision is b_{bb}=Rsin(α). This allows the factor cos(γ)=b_{bb}/R to be easily determined geometrically for any given cut shot with natural roll and no sidespin.

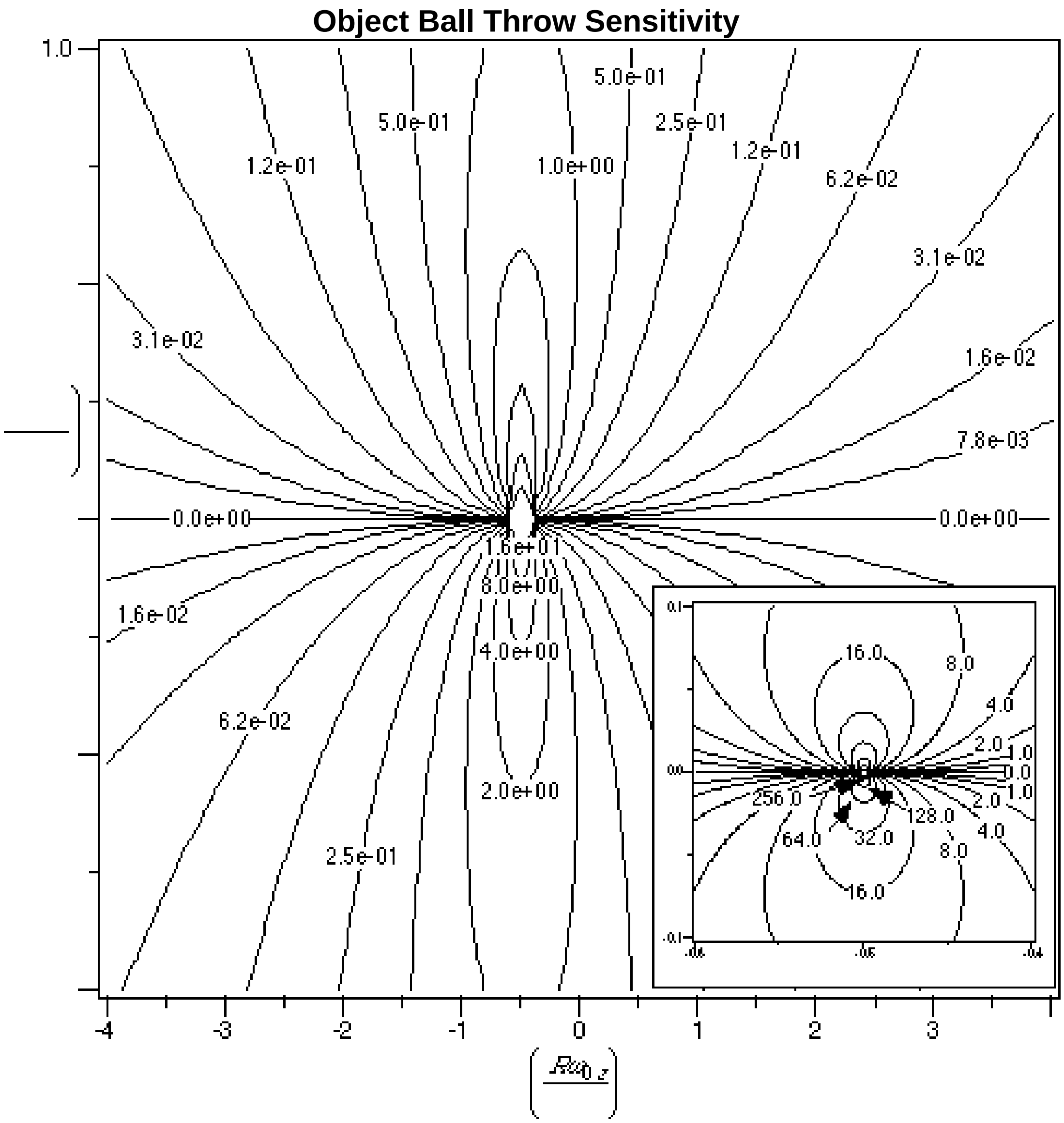

Fig. 4.5. A contour plot of the sensitivity of the object ball throw factor cos(γ) is shown as a function of the cue ball sidespin to speed ratios J_{0z}=($R\omega_{0z}/V_0$) and the topspin-draw spin to speed ratio J_{0y}=($R\omega'_{0y}/V_0$). Adjacent contours differ by a factor of two in the sensitivity function $F(\mathbf{J}_0)$. The inset figure is an expanded view of the small region near J_{0y}=0 and V_{cpy}=0.

Based on these considerations, the following procedure may be used to adjust for object ball throw for natural roll shots with no sidespin. (1) Determine μ_{bb} using the procedure in P1.6. This only needs to be done once for a given set of balls. (2) For the particular shot of interest, estimate the distance D from the object ball to the pocket; the corresponding maximum throw distance will be $\mu_{bb}D$. (3) For the zero-friction cut angle for the particular shot of interest, estimate the impact parameter for the ball-ball collision, and the ratio b_{bb}/R. (4) Multiply the maximum throw distance $\mu_{bb}D$ by the impact parameter ratio b_{bb}/R, and call the result *s*. (5) Imagine a point that is displaced by the distance *s* from the pocket target, and aim for this offset point as if there were no throw.

For an example of this procedure, assume that μ_{bb} has been determined for the set of balls as in P1.6 to be 4/72. For the shot of interest, the distance from the object ball to the pocket is 36". The maximum throw distance for this shot is (4/72)*36"=2"; that is, half the reference shot distance results in half the maximum throw distance. Suppose that the shot of interest is almost straight-in, a slight cut to the left, with $b_{bb}/R=1/4$. The offset distance is given by s=1/4*2"=1/2". Now a displaced point 1/2" to the left of the pocket center is used as a corrected aim point. This aim point is valid for either natural roll or reverse natural roll. With a little bit of practice, these estimations become second nature and may be done almost instantaneously. For other ω'_{0y} spin combinations, the offset point will be displaced from the target pocket somewhere between the maximum value of 2" (appropriate for a stun shot) and the natural roll value of 1/2", but the offset aim point will always be on the “overcut” side of the pocket center. Experienced pool players know to “cut ‘em thin to win” when the balls are sticky, and the above procedure quantifies just “how thin” is “thin” to achieve the most consistent results.

The use of sidespin also requires further adjustments to the above procedure, but this requires even more judgement on the part of the shooter. One way to adjust for sidespin is to estimate mentally the cos(γ) factor by imagining how the cue ball will be spinning at the time of contact. Replacing the cue ball with a striped ball, and practicing various combinations of topspin, draw, stun, and sidespin will help the player develop this estimation skill. In general, the offset point will always be displaced less than the maximum value determined by $\mu_{bb}D$. Of course, small μ_{bb} values mean that any errors made in the estimation of the cos(γ) factor result in smaller errors in the object ball trajectory. Sticky balls with large μ_{bb} are very challenging. One of the challenges faced by tournament players is the accurate adjustment to different sets of balls, each with different μ_{bb}, as they move from table to table in the tournament matches.

Problem 4.7: What is the resulting object ball spin ω_b due to the frictional force $\mathbf{F}(t)$?
Answer: The angular acceleration is given by the equation $\mathbf{r}\times\mathbf{F}=\mathbf{I}\,\dot{\boldsymbol{\omega}}$. Integration of the force over the contact time gives

$$\omega_b = \frac{-R}{I}\int_0^t \hat{\mathbf{i}}\times\mathbf{F}(t')dt' = \frac{-R\mu_{bb}}{I}\,\hat{\mathbf{i}}\times\left(\cos(\gamma)\hat{\mathbf{j}}+\sin(\gamma)\hat{\mathbf{k}}\right)\int_0^t F_x(t')\,dt'$$
$$= \frac{-5\mu_{bb}V_{bx}}{2R}\left(-\sin(\gamma)\hat{\mathbf{j}}+\cos(\gamma)\hat{\mathbf{k}}\right)$$

Problem 4.8: What is the relation between the natural roll spin axis and the object ball throw angle? (For simplicity, ignore the effects of the vertical friction components during the cue ball and object ball collision.)

Answer: Since both the object ball sidespin and the object ball throw angle are caused by the same frictional force, the magnitudes of these two effects are closely related. Immediately after the collision with the cue ball (and ignoring the object ball spin due to the vertical friction components), the object ball linear velocity and angular velocity vectors are given by

$$\mathbf{V}_b = V_{bx}\hat{\mathbf{i}} + V_{by}\hat{\mathbf{j}} = V_{bx}\hat{\mathbf{i}} + \mu_{bb}\cos(\gamma)V_{bx}\hat{\mathbf{j}} = V_b\hat{\mathbf{e}}_b$$

$$\omega_b = \omega_{bz}\hat{\mathbf{k}} = -\left(\frac{5\mu_{bb}\cos(\gamma)V_{bx}}{2R}\right)\hat{\mathbf{k}}$$

with $\tan(\varepsilon)=\mu_{bb}\cos(\gamma)$. After achieving natural roll, the object ball linear and angular velocity vectors are

$$\mathbf{V}_{bNR} = \tfrac{5}{7}V_b\hat{\mathbf{e}}_b$$

$$\omega_{bNR} = \omega_b + \frac{5V_b}{2R}\hat{\mathbf{e}}_b$$

where the unit vector $\hat{\mathbf{e}}_b = \hat{\mathbf{k}} \times \hat{\mathbf{e}}_b$ is the horizontal vector perpendicular to the object ball velocity. The angle of the natural roll spin axis is related to the components of the spin axes according to

$$\tan(\beta) = \frac{\omega_z}{\omega_z} = -\frac{5V_{by}}{2V_{bNR}} = -\frac{7\tan(\varepsilon)}{2\sqrt{1+\tan^2(\varepsilon)}} = -\frac{7}{2}\sin(\varepsilon)$$

For the typically small object ball throw angles, the approximate relations

$$\beta \approx -\tfrac{7}{2}\varepsilon \approx -\tfrac{7}{2}\mu_{bb}\cos(\gamma) \qquad \text{[for small } \varepsilon\text{]}$$

show that the natural roll spin axis tilt angle is about 3½ times larger in magnitude than the corresponding object ball throw angle, and that both angles are approximately linear with respect to the ball-ball friction coefficient. This axis tilt is most easily observed by viewing the rolling object ball from directly behind its path and by noting the equivalent tilt of the stationary rotational equator. The relation between the spin axis and the rotational equator is shown in Fig. 4.6. This axis tilt may be used to give the player additional feedback in adjusting the compensation for object ball throw on cut shots.

P4.7 gives the resulting object ball spin if the frictional force acts on the ball without opposition. During the collision, in order for a horizontal component of angular acceleration to occur, the ball-ball friction must act simultaneously with the ball-cloth friction. It will be assumed hereafter that the ball-cloth friction is insignificant during the collision time, and its affects will be ignored. The practical accuracy of this approximation may be estimated by the following considerations. A typical collision time is t=0.0001s, and a typical object ball velocity is V_{bx}=100in/s. The average impact force is then given by $F_{avg}=MV_{bx}/t$. The sliding frictional force of the ball on the cloth is given by $F_s=\mu_s Mg$. The ratio is given by $F_s/F_{avg}=\mu_s gt/V_{bx}$. Assuming a ball-cloth

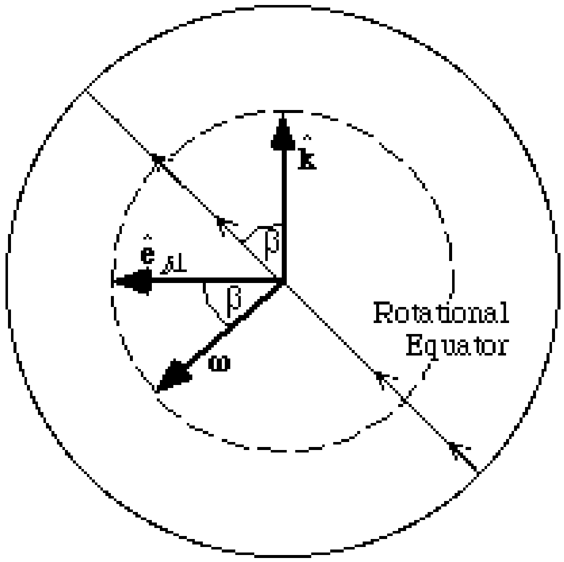

Fig. 4.6. The relation between the tilt of the spin axis ω and the rotational equator is shown pictorially as viewed from the rear of a naturally rolling ball. For an object ball, the angle β of the tilt of this axis is approximately 7⁄2 times larger in magnitude than the object ball throw angle.

sliding coefficient of friction μ_s=0.1, this ratio is F_s/F_{avg}=0.0000386. Therefore, the ball-ball frictional forces do indeed dominate the ball-cloth frictional forces during the collision.

The treatment of the vertical acceleration due to the vertical component of the frictional force is somewhat complicated. The table surface prevents any vertical acceleration in the downward direction. The weight of the ball opposes any upward frictional force, but it doesn't prevent upward acceleration. Therefore, during the contact period, if the ball is on the table surface and (F_z–Mg) is negative, resulting in a downward net force, there is no acceleration at that instant. But if (F_z–Mg) is positive, then that upward force results in vertical acceleration of the ball off the table surface. If Mg is neglible compared to a large positive F_z then the maximum vertical velocity immediately after the collision would be the same as the maximum throw velocity; the maximum angle that the object ball departs from the table surface would be the same as the maximum horizontal object ball throw angle. With the average impact force given by $F_{avg}=MV_{bx}/t$ and the downward force of gravity given by $F_{grav}=Mg$, then the ratio is given by $F_{grav}/F_{avg}=gt/V_{bx}$. For the typical shot considered in the previous paragraph, the numerical value of this ratio is F_{grav}/F_{avg}=0.000386. Therefore, the ball-ball frictional forces also dominate the gravitational forces during the collision.

Problem 4.9: A cue ball with backspin strikes an object ball straight on. Assume the gravitational force on the ball is negligible during the collision, a shot speed of 36"/*s*, and μ_{bb}=4/72 as in P1.6. What height does the object ball achieve over the table, and how far away from the starting point does it land?

Answer: The vertical velocity is given by

$$V_{bz} = \mu_{bb}\sin(\gamma)\, V_{bx}$$

For a straight on shot with backspin, sin(γ)=+1 and the entire frictional force is directed upward. V_{bz} =(4/72)36"/s=2"/s. The height of the ball trajectory above the table is given by

$$z = V_{bz}\, t - \tfrac{1}{2} g t^2 = (2\text{"}/s)t - \tfrac{1}{2}(386\text{"}/s^2)t^2$$

The maximum height is achieved when dz/dt=0. This occurs at $t_{max} = V_{bz}/g = \mu_{bb} V_{bx}/g$. The time to achieve maximum height is linear in the coefficient of friction μ_{bb} and in the shot speed V_{bx} .

$$t_{max} = V_{bz}/g = 2/386\ s = 0.00518\ s$$

The height achieved at this time is

$$z_{max} = V_{bz}\, t_{max} - \tfrac{1}{2} g t_{max}{}^2 = \frac{V_{bz}^2}{2g} = \frac{\mu_{bb}^2 V_{bx}^2}{2g}$$

$$= (2/386)\text{"} = 0.00518\text{"}$$

The maximum height achieved is proportional to the square of the coefficient of friction and to the square of the shot speed. The ball returns to the table at the time ($2t_{max}$). At this time, the horizontal distance traveled by the ball while airborne is

$$x = V_{bx}\,(2t_{max}) = \frac{2\mu_{bb} V_{bx}^2}{g}$$

$$= 36\text{"}(2)(2/386) = 0.373\text{"}$$

The horizontal distance of the jump is proportional to the coefficient of friction and to the square of the shot speed. Due to the very short times and small distances that the object ball is airborne, this jumping effect can be neglected, for the most part, during play.

One point to notice in P4.9 is that while the object ball has a vertical momentum immediately after the collision, the cue ball is constrained to the table surface. If the cue ball strikes the object ball with topspin, then it is the cue ball that leaves the table and the object ball that is constrained to the table surface. In either case, the vertical component of the linear momentum is not conserved by the balls during the collision. The reaction of the downward-directed ball is absorbed by the table. If the table had been considered to be part of the system, then linear momentum would have been conserved in the analysis. In this respect, the nonconservation of linear momentum in the vertical direction is an artifact of the formal separation between the “system” and the “surroundings” in this simple analysis.

Problem 4.10: Using the velocity and spin results from P4.2-P4.7, compute the total kinetic energy before and after the collision. Determine $E_{inelastic}$. (For simplicity, ignore the velocity and spin resulting from the vertical components of the frictional force.)

Answer: The total kinetic energy immediately before the collision is

$$T_0 = T_{0(Trans)} + T_{0(Rot)} = \tfrac{1}{2} M V_0^2 + \tfrac{1}{2} I \omega_0^2$$

The kinetic energy immediately after the collision is

$$T_f = \tfrac{1}{2} M\left(V_c^2 + V_b^2\right) + \tfrac{1}{2} I\left(\omega_c^2 + \omega_b^2\right)$$

Writing all of the friction-dependent contributions in terms of V_{by} gives

$$V_{by} = \mu_{bb} \cos(\gamma) V_{bx} = \mu_{bb} \cos(\gamma) V_0 \cos(\alpha)$$

$$\mathbf{V}_b = V_0 \cos(\alpha)\hat{\mathbf{i}} + V_{by}\hat{\mathbf{j}}$$

$$\mathbf{V}_c = \left(V_0 \sin(\alpha) - V_{by}\right)\hat{\mathbf{j}}$$

$$\omega_b = \frac{-5V_{by}}{2R}\hat{\mathbf{k}}$$

$$\omega_c = \omega_0 + \omega_b = -\omega'_{0y} \sin(\alpha)\hat{\mathbf{i}} + \omega'_{0y} \cos(\alpha)\hat{\mathbf{j}} + \left(\omega_{0z} - \frac{5V_{by}}{2R}\right)\hat{\mathbf{k}}$$

Substitution into the kinetic energy expression gives

$$T_f = T_0 + M\left(-V_{by}\left(V_0 \sin(\alpha) + R\omega_{0z}\right) + \tfrac{7}{2} V_{by}^2\right)$$

$$= T_0 + M\left(-V_{by}V_{cpy} + \tfrac{7}{2} V_{by}^2\right)$$

The kinetic energy change $E_{inelastic}$ is given by

$$E_{inelastic} = T_0 - T_f = M\left(V_{by}\left(V_0 \sin(\alpha) + R\omega_{0z}\right) - \tfrac{7}{2} V_{by}^2\right)$$

$$= M\left(V_{by}V_{cpy} - \tfrac{7}{2} V_{by}^2\right)$$

The friction allows for transfer of energy between the translational and rotational degrees of freedom, but only at a cost. This is consistent with the effect of ball-cloth friction on the kinetic energy as discussed previously. In the expressions above, V_{cpy} is the horizontal tangential component of the contact point velocity of the cue ball at the instant of collision. V_{cpy} determines the direction of the frictional force on the object ball and therefore has the same sign as V_{by}. The lowest order term in μ_{bb} in the loss of energy due to friction, $MV_{by}V_{cpy}$, is positive. The second term, which is second order in μ_{bb} and therefore in general much smaller in magnitude, is always negative.

Problem 4.11: Determine $\Delta_{elastic}$, $\Delta_{inelastic}$, and Δ_{total} in terms of V_{by}. What are these quantities when V_{cpy} =0?

Answer: From P4.10, $\Delta_{inelastic}$ is given by

$$\Delta_{inelastic} = \tfrac{2}{M} E_{inelastic} = 2V_{by}V_{cpy} - 7V_{by}^2$$

Generalizing the approach of P4.1 for arbitrary cue ball spin ω_0,

$$\Delta_{elastic} = \tfrac{2}{5} R^2\left(\omega_c^2 + \omega_b^2 - \omega_0^2\right) = \tfrac{2}{5} R^2\left(\omega_c^2 + \omega_b^2 - \left(\omega_c - \omega_b\right)\ \left(\omega_c - \omega_b\right)\right)$$

$$= \tfrac{4}{5} R^2\ \omega_c\ \ \omega_b$$

$$= -2R\omega_{0z}V_{by} + 5V_{by}^2$$

$$\Delta_{total} = \Delta_{elastic} + \Delta_{inelastic} = 2V_{by}V_0 \sin(\alpha) - 2V_{by}^2$$

In general Δ_{total} is a quadratic function of the ball-ball sliding coefficient of friction μ_{bb}.