Amateur Physics for the Amateur Pool Player

Third Edition

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Introduction

The word *amateur* is based on the Latin words *amator* (a lover) and *amare* (to love). An amateur is someone who loves what he does, and pursues it for the pleasure of the act itself. These notes are intended for the pool player who enjoys playing the game, and who enjoys understanding how things work using the language of physics. There is probably very little pool playing technique discussed in this manuscript that will be new to the experienced pool player, and likewise, there is little physics that will be new to the experienced physicist. However, there will be hopefully new pool technique for the interested physicist and new physics for the interested pool player. The tone of the presentation is not directed necessarily toward either the pool student or the physics student, but rather toward the amateur who enjoys both. The physics that is used here is not derived from first principles; it is assumed that the reader is familiar with such ideas as Newton's laws of motion, center of mass transformations, moments of inertia, linear and angular acceleration, geometry, trigonometry, and vector notation. Reference to a calculus-based introductory college level physics textbook should be sufficient to understand fully any of the physics used or mentioned in this text. *The Feynman Lectures on Physics* (Vol. 1) is one such text that the reader will find enjoyable.

This discussion is divided into five sections. Section 1 discusses the equipment (balls, tables, cue sticks, cue tip, cloth) and some of its associated properties (various friction coefficients, forces, moments of inertia), section 2 discusses the concept of natural roll, section 3 discusses the cue tip and cue ball impact, section 4 discusses collisions between balls, and section 5 discusses the use of statistical methods. Each section includes some general discussion and specific problems (along with their solutions). Some exercises are also given along the way; it is intended for the reader to experiment on a pool table with some of the techniques that have been discussed.

1. Properties of the Equipment

Pool, billiard, and snooker balls are uniform spheres of, usually, a phenolic resin type of plastic. Older balls have been made of clay, ivory, wood, and other materials. On coin-operated tables, the cue ball is sometimes larger and heavier than the other balls; otherwise, all the balls in a set are the same size and weight. Standard pool balls are $2\frac{1}{4}$" in diameter, snooker balls are either of two sizes, $2\frac{1}{8}$" or $2\frac{1}{16}$", and carom billiard balls are one of three sizes, $2\frac{27}{64}$", $2\frac{3}{8}$", or $2\frac{7}{16}$". Tolerances in all cases are ±0.005". Pool balls weigh 5.5 to 6*oz*, snooker balls weigh 5 to 5.5*oz*, and billiard balls weigh 7 to 7.5*oz*.

Problem 1.1: What is the volume of a pool ball in terms of its radius *R*?
Answer: In spherical coordinates, the volume of a sphere is given by

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr d\theta d\varphi = \left(\tfrac{1}{3}R^3\right)(2)(2\pi) = \tfrac{4}{3}\pi R^3$$

where *R* is the radius of the ball. Assuming that the ball is a perfect sphere, the minimum radius is R_{min}=1.1225" and the maximum radius is R_{max}=1.1275". The volume of a standard pool ball is between $\frac{4}{3}\pi R_{\min}^3$ =5.924in^3=97.08cm^3 and $\frac{4}{3}\pi R_{\max}^3$ =6.004in^3=98.39cm^3.

Problem 1.2: In order to satisfy the size and weight limits, what is the density range of the ball material in units of g/cm^3?
Answer: The density is the mass divided by the volume, $\rho=M/V$. The minimum mass is 5.5*oz*(28.35*g*/*oz*)=155.9*g*, and the maximum mass is 6.0*oz*(28.35*g*/*oz*)=170.1*g*. The minimum density is $\rho_{min}=M_{min}/V_{max}$=1.559*g*/98.39$cm^3$=1.58$g/cm^3$ and the maximum density is $\rho_{max}=M_{max}/V_{min}$ =170.1*g*/97.08cm^3=1.75g/cm^3. For comparison, the density of water at room termperature is 0.997g/cm^3 , a saturated sucrose (table sugar) solution is 1.44g/cm^3, a saturated cesium chloride solution is 1.89g/cm^3, and the density of mercury is 13.6g/cm^3, so a pool ball should easily sink in water, slowly sink in the sugar solution, barely float in the cesium chloride solution, and easily float in mercury.

The inertia tensor of a rigid body is defined as the elements of the 3 by 3 matrix

$$I_{ij} = \int_V \rho(\mathbf{r}) \left[\delta_{ij} \sum_k r_k^2 - r_i r_j \right] dv$$

where the components of the vector **r**=(*x*,*y*,*z*) are the cartesian coordinates. For a uniform sphere, $\rho(\mathbf{r})=\rho$ is a constant for $r<R$ and is the density of the ball material. The mass of the ball is $M = \rho V = \frac{4}{3}\rho\pi R^3$.

Problem 1.3: Determine the inertia tensor for a ball in terms of *M* and *R*.

Answer: Taking the moment of inertia about the *x*-axis gives

$$I_{xx} = \int_V \rho(\mathbf{r})\left(z^2 + y^2\right)dv = S_{zz} + S_{yy} = 2S_{zz}$$

It is interesting to notice that the moment of inertia about the *x*-axis, for example as given above, depends only on how the mass of the object is distributed along the *z*- and *y*-axes. Some thoughtful reflection will reveal that, for the coordinate axes origin taken to be the center of the sphere, the z^2 integral S_{zz} is the same as the y^2 integral S_{yy}, so only one integral really needs to be done as indicated in the last equality above. In fact, $S_{xx}=S_{yy}=S_{zz}$ since for a sphere, the choice of axis is completely arbitrary. Using $z = r\cos(\theta)$, $x = r\sin(\theta)\cos(\varphi)$, and $y = r\sin(\theta)\sin(\varphi)$ allows these integrals to be written in polar coordinates. Taking S_{zz} for example gives

$$S_{zz} = \rho\int_0^R r^4 dr \int_0^\pi \cos^2\theta\sin\theta d\theta \int_0^{2\pi} d\varphi = \rho\left(\tfrac{1}{5}R^5\right)\left(\tfrac{2}{3}\right)(2\pi) = \tfrac{1}{5}MR^2$$

The moment of inertia about any axis is twice this value, giving $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}MR^2$. It may also be seen that the off-diagonal elements of the inertia tensor are all zero. This means that any choice of orthogonal coordinate axes is formally equivalent to any other, and any such choice corresponds to the *principle axes*. For other rigid bodies, the off-diagonal elements are generally nonzero, and only a special choice of the coordinate axes will result in a diagonal inertia tensor. Written as a matrix, the inertia tensor is

$$\mathbf{I} = \frac{2MR^2}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An important property of this inertia tensor is that its product with any vector ω is simply a scaling of that vector, the direction does not change: $\mathbf{I}\omega = \left(\frac{2}{5}MR^2\right)\omega$.

The kinetic energy of a ball consists of two parts, translational and rotational. The translational kinetic energy is given by $T_{(Trans)}={}^1/_2MV^2$, where *V* is the velocity of the center of mass of the ball. The mass of the ball, *M*, is the proportionality constant between the velocity squared and the energy. The rotational kinetic energy about a principle axis is given by the similar equation $T_{(Rot)}={}^1/_2I\omega^2$, where ω is the angular velocity, for example in radians per second. Therefore the moment of inertia, *I*, is the proportionality constant between the angular velocity squared and the rotational kinetic energy. The most general equation for the rotational energy of a rigid body is $T_{(Rot)}={}^1/_2\omega\ \mathbf{I}\ \omega$, in which ω is the angular velocity about each axis, **I** is the 3 by 3 inertia tensor, and the dot implies the appropriate matrix-vector or vector-vector product. The quantity **L**=**I**ω is the rotational angular momentum about the center of mass, and the simple form for **I** given above means that for a pool ball the angular momentum is always aligned with the angular rotation. The rotational energy may then be written as $T_{(Rot)} = \left(\frac{1}{5}MR^2\right)\omega\ \omega = \left(\frac{1}{5}MR^2\right)|\omega|^2$. The freedom of axes choice for a uniform sphere will often allow the problem at hand to be simplified to only a single rotation axis, in which case the simple scalar equation may be used

When a force is applied to a rigid body, such as a ball, the velocity of the center of mass changes according to the equation $\mathbf{F} = M\dot{\mathbf{V}}$, and the angular velocity changes according to the equation $\mathbf{r} \times \mathbf{F} = \mathbf{I}\,\dot{\omega}$. When a single principle rotational axis is considered, the latter equation reduces to the simpler $r\sin(\theta)|F| = I\,\dot{\omega}$, where θ is the angle between the vectors **r** and **F**, with magnitudes *r* and |*F*| respectively. ω is in the direction perpendicular to the plane defined by the two vectors **r** and **F**, and aligned, by convention, with the right-hand-rule (*i.e.* when the fingers of the right hand curl in the direction that rotates **r** into **F**, then the thumb points along the direction of positive ω; other analytic expressions for the vector cross product will also be used in this discussion, but the right-hand-rule provides a useful and intuitive defintion.) The vector **r** points from the center of mass of the ball to the point on the surface of the ball at which the force is applied. In these equations, $\dot{\mathbf{V}} \equiv \frac{d\mathbf{V}}{dt}$ is the linear acceleration along each coordinate axis and $\dot{\omega} \equiv \frac{d\omega}{dt}$ is the angular acceleration around each coordinate axis. The similarities in the relations between the force and the mass *M* for the linear acceleration and between the force and the moment of inertia *I* for the rotational acceleration are again seen. The *r*sin(θ) factor shows how the angular acceleration depends on the direction of the force. When the force is applied directly toward the center of mass of the ball, then the sin(θ) factor is zero and there is no angular acceleration; it is only when the force is applied in a direction askew from the center of the ball that angular acceleration occurs.

A force is required to rub two objects together. If the two objects are pressed together with a normal force F_N, and a sideways force of magnitude F_f causes the two objects to slip against each other without acceleration, then the *coefficient of sliding friction* is defined as $\mu_{(sliding)}=F_f/F_N$. To a good approximation, the coefficient of friction between two surfaces is a constant, independent of the forces and independent of the speeds of the two sliding objects. A small coefficient of friction is associated with slippery object pairs, and a large coefficient of friction is associated with sticky object pairs. There is also a *static* coefficient of friction. Static friction is defined in a similar manner to sliding friction, but it applies to two surfaces that are at rest. For a given pair of surfaces, the static coefficient of friction is larger than the sliding coefficient, although for some surface pairs they are very close in value.

There are several frictional forces that are important in pool. The first is the sliding friction of a ball on the cloth, F_s. $F_s=\mu_{(sliding)}W$ where *W* is the weight of the ball ($F_N=W=Mg$ where *g* is the acceleration of gravity). Since the ball weight and the coefficient of friction are constants for a given ball and for a given table, the frictional force of a sliding ball is a constant. The magnitude of the frictional force does not depend on the velocity of the ball or upon ω for the ball as long as the ball is sliding on the cloth. The direction of this force does depend on the ball velocity and ω, and this will be examined in more detail in the following discussions. If the ball is not sliding on the cloth (e.g. the ball is at rest, or the ball is rolling smoothly without slipping on the cloth surface), then there is no sliding frictional force.

It is interesting to consider the nature of the cause of a sliding frictional force. At a microscopic level, the atoms in the molecules of one surface are attracted to those of the other surface. As the object slides forward, new interactions, or bonds, are formed in the forward direction, maintained momentarily, and then broken as the individual atoms are pulled apart. However, it is not directly these bonds that cause the friction. The reason is that the same kinetic energy is lost in forming the bond as is gained back again when it breaks, and there is no net change of energy due to the forming and breaking of these bonds as the surfaces slide across each other. But for the small amount of time that the individual atoms interact, vibrational energy of the surface molecules is transferred to the other molecules in the bulk of the objects. (Energy is also transferred in the opposite direction, but at a much smaller rate; the net energy flow is from the surface atoms to the bulk atoms, a consequence of the second law of thermodynamics.) The result of this energy transfer is that translational kinetic energy is transformed into vibrations of the molecules of the bulk materials, or in other words, into heat and sound. From this point of view of a physicist, it might be said that it is the heat and sound that cause the frictional force; this is somewhat the opposite of the layman's point of view, namely, that friction causes the heat.

Problem 1.4: A block slides down an inclined plane without acceleration; what is the relation between μ and the angle of the slope of the plane?
Answer: The downward force is the weight of the object $W=Mg$. The component of this force normal to the plane surface is $F_N=W\cos(\alpha)$ where α is the angle of incline. The component of the downward force tangent to the surface of the plane is $F_t=W\sin(\alpha)$. This force is directed down the incline, accelerating the object, and it is opposed by the frictional force which is directed uphill. Since the object is sliding without acceleration, all of this tangential force is balanced exactly by the frictional force, $F_S=-F_t$. The coefficient of friction is then given by $\mu=F_t/F_N=\tan(\alpha)$. This relation between slope and the coefficient of friction is so fundamental that it is sometimes taken as a *de facto* defintion.

A sliding block provides a simple conceptual model for understanding several other aspects of sliding friction. Consider a sliding block of mass M on a level surface with a sliding coefficient of friction μ. The downward force of the block is the weight of the block, $W=Mg$, and this force is exactly opposed by an upward force of the surface; this means that the block does not accelerate in the vertical direction. The horizontal force is constant in magnitude, $|F_s|=\mu W=\mu Mg$ and the direction of this force is opposite to the velocity which is taken to define the positive direction. This frictional force slows down the sliding block according to the equation $\dot{V}=-\mu g$ where the minus sign is due to the direction of the force. It is interesting that this equation does not depend on the block mass; several equations of motion in the following discussions will be similarly independent of the ball masses. Integration over time gives $V(t)=V_0-\mu gt$ where V_0 is the initial velocity at t=0. Of course, this equation is valid only as long as the block is sliding. Integration again over time gives the distance x as a function of time as $x=V_0t-\frac{1}{2}\mu gt^2$ where the distance is measured from the starting point.

Since the block is slowing down, kinetic energy is not conserved in this process. This is a *dissipative* system, not a *conservative* system. How does the kinetic energy depend on time and distance? Substitution of $V(t)$ above gives

$$T = \tfrac{1}{2}MV^2 = \tfrac{1}{2}M(V_0^2 - 2V_0\mu gt + \mu^2 g^2 t^2)$$

$$=T_0 - \mu Mgx.$$

Kinetic energy is lost as a linear function of the distance and a quadratic function of time. When the block slides to rest, T=0, the initial energy and total sliding distance d are simply related as $T_0=\mu Mgd$. If the initial energy of the block were doubled, then the distance that the block slides before coming to rest would also double. However, if the initial velocity were doubled, then the final distance would increase by a factor of four. Note also that for a given initial energy T_0, if the coefficient of friction were to increase, then the total sliding distance must decrease, and if the coefficient of friction were to decrease, then the total sliding distance must increase. A related quantity of interest is the *power dissipation*, defined as $\dot{T} \equiv \frac{dT}{dt}$. From the quadratic time function, or using the chain rule $\dot{T} = \frac{dT}{dx}\frac{dx}{dt}$, the power dissipation for a sliding block is seen to be $\dot{T}=-\mu MgV$. The treatment of frictional forces for a sliding block are relatively simple; the somewhat more complicated situations for a billiard ball sliding on a table and for two colliding billiard balls are treated in the following sections.

How can the coefficient of friction be measured? There are several possibilities, depending on the equipment available with which to make measurements or on the data available. (1) One method would be to attach a measuring scale to the block, and simply measure the force required to slide the block on the surface without acceleration; this force divided by the weight of the block would give directly the coefficient μ. (2) If the surface can be held at an arbitrary slope, then μ can be determined as in P1.4. This may not be always practical (for example if the surface is a heavy billiard table). (3) If the velocity or the energy could be measured accurately at two points in a given trajectory, then the equation $T=T_0-\mu Mgx$ at these two points could be used to determine T_0 and the product μMg. An independent determination of the weight Mg would then allow μ to be determined. However, velocities are relatively difficult to measure, so this also may not be practical. (4) Suppose that the block slides a distance d in time t_d before coming to rest. Then the initial velocity was $V_0=\mu g t_d$. Substitution of this into the quadratic distance equation gives $\mu = d/(\frac{1}{2} g t_d^2)$. Of course, this is not an exhaustive list of possibilities, and many other schemes could be devised based on preparation of the initial velocity or trajectory measurements of various types.

A second force is the rolling resistance of a ball on the cloth. This is not, strictly speaking, a sliding frictional force since it does not invlove sliding surfaces, but the formal treatment of this force is similar to the above sliding frictional force. A detailed examination of the forces involved in this situation will be postponed until the next section. For the present discussion, this rolling resistance will be modeled as a ball rolling uphill on an inclined plane. This is a conservative model. The dissipative energy loss of an actual billiard ball is then considered to be analogous to the energy loss of the model ball in the conservative gravity field. Because this model is a conservative system, it is possible to determine the equations of motion of the ball without detailed consideration of the forces (which may not be intuitively obvious for this situation).

For an incline of slope α, the height above the starting point is given by $h=s\sin(\alpha)$, where s is the distance up the incline from the starting point. The potential energy is then by given as a function of s by $U(s)=Mgh=sMg\sin(\alpha)$. In this model it is assumed that there is no energy dissipation through heat. The total energy $E=T+U$ is a constant, so any kinetic energy lost by the ball is transferred to potential energy in the gravity field. This gives the relation $T(s)=T_0-\sin(\alpha)sMg$, where $T_0=E$ is the initial energy of the rolling ball at the bottom of the incline. It is now seen that the kinetic energy for a ball rolling on an incline obeys the same equation as for the sliding block, but with the incline slope, corresponding to $\sin(\alpha)$, assuming the role of the sliding coefficient of friction of the block. However, in the case of a rolling ball, the kinetic energy expression is more complicated, and this, along with the examination of the associated forces, is discussed in more detail in the following section. Using the chain rule expression, the power dissipation for the ball rolling up an incline is given by $\dot{T} = \frac{dT}{ds}\frac{ds}{dt} = -\sin(\alpha)MgV$, where V is determined by the speed parallel to the incline. If, for some reason, it were not possible to measure the slope of the incline, it could be determined indirectly by measuring the $\sin(\alpha)$ factor in the above equations in the same manner that the sliding coefficient of friction μ can be measured for a sliding block.

The connection between an actual ball rolling on a level table and this model problem may be justified by considering the rolling ball at a microscopic level. The nature of the effective frictional force arises in part from the compression of the cloth fibers as the ball rolls past. Once compressed, they do not rebound immediately as the ball passes; if they did, then there would be no energy lost in this manner by the rolling ball. The energy lost by this irreversible compression of the fibers slows the rolling ball. Energy of the rolling ball is also lost to vibrations of the ball and table, and eventually to the increased temperature of the surroundings. As the ball rolls forward an infinitesimal amount, it rolls also uphill on the cloth, losing a small amount of kinetic energy. But the cloth cannot support the ball weight, so it compresses the fibers. This transfers the potential energy from the gravity field into the spring constants of these compressed fibers. As the ball continues to roll, the fibers remain compressed for a small time, and this time lag prevents the potential energy stored in the fibers from being returned to the ball kinetic energy. The horizontal distance that the ball rolls on the table can be measured, but the effective height that it would have risen if the cloth fibers had not compressed cannot be measured directly. Therefore, the effective slope $\sin(\alpha)$, which may be associated with an effective rolling coefficient of friction $\mu^{eff}_{(rolling)}$, must be determined indirectly.

Consider a ball rolling a distance d on a table in time t before coming to a stop. At this time, an effective force is assumed of the form $F_r = \mu^{eff}_{(rolling)} Mg$ that opposes the rolling ball. Newton's equation $F_r = M\dot{V}$ may be rewritten as $\mu^{eff}_{(rolling)} g = -\dot{V}$. Integration over time results in $\mu^{eff}_{(rolling)} gt = V_0 - V$ where V_0 is the initial velocity. Integration over time again gives $\frac{1}{2}\mu^{eff}_{(rolling)} gt^2 = V_0 t - d$. The final velocity is zero when $V_0 = \mu^{eff}_{(rolling)} gt$ and this may be used to eliminate V_0 from the distance equation. The effective coefficient of friction for the rolling ball may then be determined from the equation

$$\mu^{eff}_{(rolling)} = \frac{d}{\frac{1}{2} g t^2}$$

The ball mass does not appear in this relation. The dimensionless quantity *table speed* is defined as $1/\mu^{eff}_{(rolling)}$ and is similarly independent of ball mass. With this definition of table speed, a very slow table is in the range of 50-70. Normal table speed is 80-100. A very fast pool table might have a speed higher than 120. The cloth on a billiard table is usually finer and smoother than that on a pool table, and a fast billiard table might have a speed over 150. The force due to rolling resistance is much smaller than that due to sliding friction.

The sliding frictional force and the rolling frictional force of a ball on a table are independent quantities. Consider for example a ball on a hard rubber surface; the sliding friction would be very large, while the rolling resistance would be relatively small. Alternatively, consider a ball on a Teflon surface with a soft backing; the sliding friction would be relatively very small, while the rolling resistance would be relatively large. The uniformity of billiard cloth material limits the range of extremes that are encountered in practice. The official BCA (Billiard Congress of America) rules specify a billiard cloth that is predominantly wool. The PBTA (Professional Billiard Tour Association) requirements are even more specific, and detail a brand and type of billiard cloth, namely Simonis 860; although this is partly a matter of sponsorship, it may be noted that this is a relatively fast pool table cloth that results typically in table speeds of 100 to 130 when newly installed.

Problem 1.5: A ball is lagged perfectly on a standard 9' pool table and it is observed that the ball travels from the foot cushion to the head cushion in 7.00 seconds. What is the table speed? What was the initial velocity of the ball as it left the last cushion?
Answer: The playing area of a standard 9' pool table is 50" by 100". After accounting for the ball width, the center of the ball travels (100"–2.25")=97.75" between cushions. The acceleration due to gravity is g=386 in/s^2. The table speed is

$$\text{TableSpeed} = \frac{1}{\mu^{eff}_{(rolling)}} = \frac{\frac{1}{2}gt^2}{d} = \frac{0.5 \quad 386\left(\frac{in}{s^2}\right) \ t^2}{97.75(in)} = 1.97 \ t^2_{sec}$$

$$= 1.97 \ \left(7.00^2\right) = 96.7$$

This is a fairly fast pool table. It is customary to approximate the $g/(2d)$=1.97 factor as 2.0 on a 9' table. The table speed may then be estimated simply as $2t^2$ where the time is measured in seconds. For playing purposes, it is usually unimportant to know the table speed to more than 2 significant figures. The velocity after the last cushion was

$$V_0 = \mu^{eff}_{(rolling)} gt = \frac{2d}{t} = \frac{2(97.75in)}{7.0s} = 27.9\left(\frac{in}{s}\right).$$

The initial velocity is seen to be twice the time-average velocity, which is given by d/t.

Exercise 1.1: Measure the table speed of some of the tables on which you play regularly. Rather than try to lag a ball perfectly, set up a ramp with cue sticks, and adjust the height of the ramp and initial ball placement so that the ball rebounds off the foot cushion and stops just before touching the head cushion. Disregard the small time it takes for the ball to achieve natural roll after impact with the foot cushion. Take the average time for several rolls in order to account for timing inaccuracies.

A third important frictional force is that between two colliding balls. The forces between two balls change during the collision. The collision time is very short, so these forces can be very large in order to transfer energy from one ball to another during a collision. The frictional forces act in a direction tangential to the surface of the ball at the point of contact between the balls. This is shown schematically in Fig. 1.1. The linear forces that accelerate the balls are directed between the ball centers. The resultant force on a ball is the sum of these two vector forces. That velocity component of a ball due to the tangential frictional forces is called either *collision induced throw* or *spin induced throw*, depending on the spinning condition of the balls and on the cut angles involved. When two balls slide against each other, both balls are accelerated by frictional forces. The frictional force vector that accelerates one ball is exactly opposite to that which accelerates the other ball. Note however that the angular acceleration due to the frictional forces has the same sign on both balls, due to the fact that the opposing forces are applied to the front of one ball but to the back of the other. As before, to a good approximation the frictional force is independent of the speed at which the two surfaces slide against each other. The force is constant unless the spinning balls "lock" against each other (as two interlocked gears), at which time the sliding frictional force vanishes.

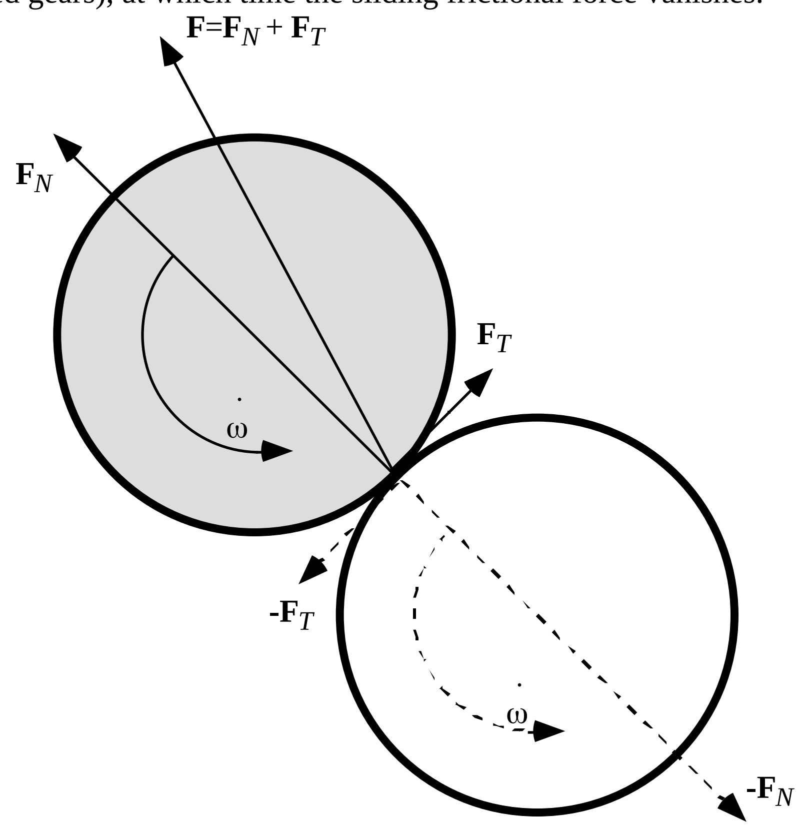

Fig. 1.1. The normal forces $\mathbf{F}_N$, tangential forces due to sliding friction $\mathbf{F}_T$, the resulting total force **F**, and the angular acceleration $\dot{\omega}$ are shown schematically for two colliding balls. The magnitudes of the forces change during the collision, but the ratio of the tangential and normal forces are constant and are determined by the coefficient of friction. The magnitude of the tangential forces are shown greatly exaggerated. Note that although the tangential forces acting on the two balls exactly oppose each other, the resulting angular accelerations have the same sign.

Problem 1.6: Two object balls are frozen together and aligned straight toward the foot cushion exactly toward a marked spot. The nearest ball is 72" away from the cushion. The farthest ball from the cushion is hit at an angle with the cue ball. The object ball is observed to miss the point on the cushion by 4". Assuming that this collision induced throw is due to friction, what is the coefficient of friction for these two balls?
Answer: $\mathbf{F}_N$ is directed toward the marked spot, and $\mathbf{F}_T$ is perpendicular as in Fig. 1.1. The resultant velocity is parallel to the total force vector. The coefficient of friction is related to the angle of throw α by

$$\tan(\alpha) = \frac{F_T}{F_N} = \mu = \frac{V_T}{V_N} = \frac{D_T}{D_N}$$

Substitution of the appropriate distances gives the coefficient of friction as

$$\mu = \frac{4''}{72''} = 0.0556$$

Exercise 1.2: Measure the collision induced throw angle for several sets of balls at pool rooms where you play regularly. Generally, if the balls are worn or dirty, they will have a high coefficient of friction, and if they are new or polished, they will have a low coefficient of friction. Smear some chalk on the contact point between the frozen balls, and an increased coefficient of friction should be observed. Smear some talcum powder on the contact point, and a smaller coefficient of friction should be seen. Place a drop of water (or spit) on the contact point and the coefficient of friction will become essentially zero. Correcting for collision induced throw is one of the challenging aspects of playing with different sets of balls in tournaments, and of playing at different pool rooms.

A fourth frictional force is the static friction between the cue tip and the cue ball. The cue tip must not slide on the cue ball. If this occurs unintentionally, then a miscue results and the cue ball behaves unpredictably; if the cue tip slides intentionally against the cue ball, then an illegal "push shot" has occurred. The static frictional force is related to the normal force and to the static coefficient of friction by the relation $\mu_{static}=F_T/F_N$ where F_T is the minimum force required to cause the cue tip to slide on the surface of the cue ball.

Problem 1.7: For a particular cue tip, it is observed that miscues begin to occur when the cue tip contacts the cue ball at a height halfway between the center and the top of the cue ball. What is the static coefficient of friction between the cue tip and the cue ball? If the static coefficient of friction is 1.0, what is the displacement at which miscues begin to occur?

Answer: Refer to Fig. 1.2. The slope of the cue ball at the point of contact. is determined by

$$\cot(\alpha) = \frac{\left(\frac{b}{R}\right)}{\sqrt{1-\left(\frac{b}{R}\right)^2}}$$

where b is the displacement away from the center. When the force F is applied to the cue ball in a horizontal direction, this may be written as a sum of the normal force toward the center of the cue ball $F_N=F\sin(\alpha)$, and the tangential frictional force with magnitude $F_T=F\cos(\alpha)$. The coefficient of friction and the maximum displacement are related by

$$\mu_{static} = \frac{F_T}{F_N} = \cot(\alpha) = \frac{\left(\frac{b_{max}}{R}\right)}{\sqrt{1 - \left(\frac{b_{max}}{R}\right)^2}}$$

$$\left(\frac{b_{max}}{R}\right) = \frac{\mu_{static}}{\sqrt{1 + \mu_{static}^2}}$$

For $b_{max}/R=½$,

$$\mu_{static} = \frac{½}{\sqrt{1 - ¼}} = \frac{1}{\sqrt{3}} = .577$$

For $\mu_{static}=1.0$,

$$\left(\frac{b_{max}}{R}\right) = \frac{1}{\sqrt{2}} = .707$$

As seen for these two cases, a higher coefficient of friction allows the cue tip to contact the cue ball at larger displacements without miscuing.

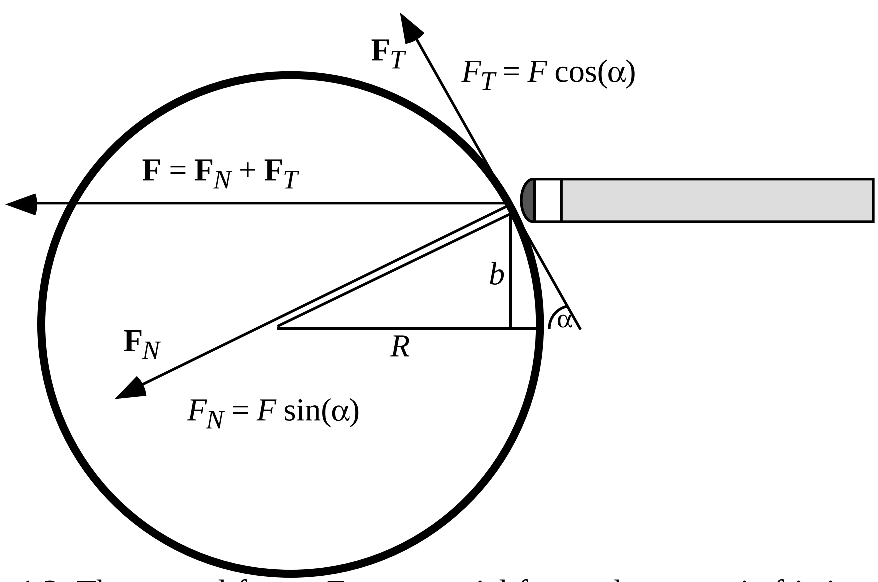

Fig. 1.2. The normal forces F_N, tangential forces due to static friction F_T, and the resulting total force F for contact between the cue tip and cue ball are shown schematically. The magnitudes of the forces change during the collision, but the ratio of the tangential and normal forces are constant and are determined by the impact point and limited by the the static coefficient of friction.

Exercise 1.3: Determine the static coefficient of friction between your cue tip and a cue ball. Instead of determining the point of miscue (as in P1.7), hold a ball against a cushion and stand the cue shaft vertically on the ball. Estimate the distance away from the center ball, and use the equation in P1.7 to determine μ_{static}. Wipe the cue tip clean, removing all chalk, and a smaller coefficient of friction should be observed. Experiment with different kinds of chalk and with different tip conditions. Note that it is the displacement of the actual contact point of the cue tip that should be measured, and not the displacement of the cue shaft edge.

2. Slide and Natural Roll

Suppose that at some time a ball is known to have some (center of mass) translational velocity and some spin (about the center of mass). For simplicity, assume that the spin axis is horizontal and is perpendicular to the translational velocity (*i.e.* the ball has straight topspin or draw; e.g. $\mathbf{V}=V\hat{\mathbf{i}}$ and $\omega=\omega\hat{\mathbf{j}}$). As the ball slides on the cloth on the table, the friction between the ball and cloth will cause both the translational and angular velocity to change. This force will act to accelerate the ball, that is, to increase or decrease the velocity, until an equilibrium situation occurs in which the translational and angular velocities "match" each other, at which time the sliding frictional force becomes zero. This is the *natural roll* (also called *normal roll, smooth roll,* or *rolling without slipping*) situation. Over a small time dt, the distance traveled by the ball will be Vdt, and the outside surface of the ball will roll a distance $R\omega dt$ relative to the ball center of mass. Therefore, this "matching" occurs when $V=R\omega$.

The natural roll condition is important to examine because the speed and spin of a sliding ball are always being forced toward the natural roll condition by the sliding friction, and once achieved, natural roll is maintained by the ball until it collides with another ball or cushion or rolls to a stop.

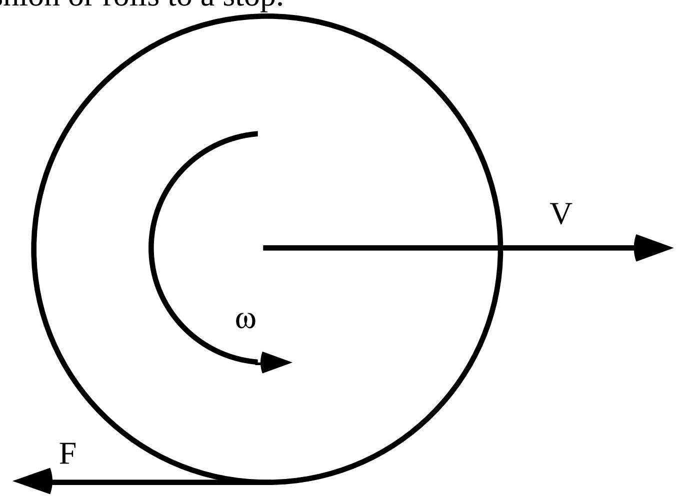

Fig. 2.1. The linear velocity V, angular velocity ω, and corresponding frictional force F are shown schematically for a backspin shot. V is positive, whereas F and ω are taken to be negative as shown.

Kinetic energy is not conserved during the equilibration period as the sliding ball approaches the natural roll condition. This is easy to see in the case in which the translational velocity and angular velocity oppose each other, as in a backspin shot depicted in Fig. 2.1. (Positive ω is taken to be in the clockwise direction in Fig. 2.1.) In a backspin shot, the initial frictional force acts to both slow down the ball and to decrease the magnitude of the spin, clearly decreasing simultaneously both types of kinetic energy.

A useful concept to introduce in this discussion is the spin/speed ratio ω/V. In some situations, a more useful quantity is the dimensionless ratio $J=(R\omega/V)$; for the above backspin shot, this is the ratio of velocity at a point on a ball on the rotational equator that is due to the spin to the velocity of the center of mass of the ball. In situations in which several spin components are examined simultaneously, the dimensionless vector quantity $\mathbf{J}=(J_x,J_y,J_z)=R\omega/V$ is useful. As discussed above $J_y=+1$ corresponds to the natural roll condition when the velocity is directed along the x-axis.

The frictional force acts on the very bottom point of the ball, where the ball touches the cloth, and it points in a horizontal direction. The force acts to accelerate the ball according to the equation $\mathbf{F} = M\dot{\mathbf{V}}$. Integrated over some time period, this gives a change of momentum

$$\mathbf{F}t = M(\mathbf{V} - \mathbf{V}_0).$$

in which $\mathbf{V}_0$ is the initial velocity vector. Note that since **F** and **V** point in opposite directions in a backspin shot; the ball slows down over time. When **F** and **V** point in the same direction, e.g. a ball over-spinning with topspin, the ball speeds up over time. In the case depicted in Fig. 2.1, this equation simplifies to

$$|F|t=-M(V-V_0)$$

or, after eliminating the mass from both side of the equation and introducing the ball-cloth sliding coefficient of friction,

$$\mu gt =-(V-V_0)$$

where the sign of the right hand sides results from the fact that the velocity and force vectors point in opposite directions. (In the general case for positive V_0, $F>0$ when $R\omega_0>V_0$, and $F<0$ when $R\omega_0<V_0$ or in other words, F and $(J-1)$ have the same sign.)

The angular velocity of the sliding ball changes according to the equation $\mathbf{r} \times \mathbf{F} = \mathbf{I}\,\dot{\omega}$. For the backspin shot, $\mathbf{r}=-R\hat{\mathbf{k}}$, $\mathbf{F}=-|F|\,\hat{\mathbf{i}}$, and $\dot{\omega} = \dot{\omega}\hat{\mathbf{j}}$. In this situation, this equation simplifies to $R|F| = I\dot{\omega}$. Integrated over some time period, this gives

$$R|F|t = I(\omega - \omega_0).$$

Note in Fig. 2.1 that for a backspin shot the frictional force is acting to increase the angular velocity from an initial negative value to a final positive value. If the cue ball contacts an object ball while the angular velocity is still negative, this is called a *draw shot*. If all the spin is removed by the cloth friction and the ball is spinning neither forward nor backward upon impact with an object ball, this is called a *stun shot*. If forward roll, or in particular natural roll, is achieved prior to collision, this is called a *drag shot*. As shown in the above equation, it is the initial angular velocity, the sliding friction between the ball and the cloth, and the time before the collision that distinguishes these three shots.

Problem 2.1: What is the relation between linear and angular velocity for a sliding ball?
Answer: Eliminating the common $|F|t$ from the above two expressions gives

$$\frac{I}{R}(\omega - \omega_0) = -M(V - V_0) \quad .$$

Using the previous expression for I for a ball results in

$$V = V_0 - \tfrac{2}{5}R(\omega - \omega_0) \quad .$$

This expression is valid at any time the ball is sliding on the cloth. Although derived specifically for the backspin shot, this expression is valid for any frictional force. Note that for the backspin shot, V decreases as ω increases, and for the over-top-spin situation, V increases as ω decreases. This shows that the relation between linear and angular velocity does not depend on the ball mass or on the ball-cloth sliding coefficient of friction.

Problem 2.2: Determine the final linear velocity of a ball after natural roll is achieved as a function of initial linear and angular velocities.

Answer: Natural roll is achieved when the linear and angular velocities equilibrate. Substituting $V=R\omega$ in the expression from P2.1 gives

$$V_{NR} = \frac{5}{7} V_0 + \frac{2}{7} R\omega_0$$

Note that if the initial angular velocity were zero, then the sliding ball would eventually slow down to $^5/_7$ of its initial velocity. If the initial angular velocity matched exactly the initial linear velocity, $V_0=R\omega_0$, then the linear velocity would remain unchanged. If the initial angular velocity is negative, as for a drag shot, then the final linear velocity is even less than $^5/_7$ of the initial velocity; for example, if the initial angular velocity is equal to natural roll, but in the opposite direction, $V_0=-R\omega_0$, then the final velocity is $^3/_7$ of the initial velocity. If the initial spin is very large and negative, then the final natural roll velocity will be negative; this can occur in masse shots, or in situations involving collisions with other balls. Note that the natural roll velocity does not depend on the ball-cloth friction or the ball mass.

Exercise 2.1: Experiment with the drag shot. Use a striped ball in place of the cue ball so that the spin is easily observed. Strike the "cue" ball below center. Observe how the ball initially spins backward. The cloth friction slows this backspin until at some point the ball is not rotating at all, but is simply sliding across the table. Beyond this point the ball begins rolling forward. At some point all sliding stops, and the ball achieves natural roll. During all of the time that the ball is sliding on the cloth, the speed of the ball is decreasing. If you have a video camera, record some of these shots and play them back in slow motion. The drag shot is useful when playing on dirty or unlevel tables, and a low-speed impact between the cue ball and object ball is required for position. The initial high speed of the cue ball reduces the effect of the unlevel table, and only at the very end after natural roll is achieved and the velocity is reduced to about $^3/_7$ of the initial velocity, does the impact occur. The average velocity of the cue ball is about $^5/_7$ of the initial velocity, which means that the effect of the unlevel table has been reduced by about $^2/_7$ or 29% from the case where natural roll is achieved immediately.

Exercise 2.2: Experiment with a stun shot. A stun shot is when the cue ball has zero angular velocity about the horizontal axis upon contact with an object ball or cushion.

Set up a straight-in shot with an object ball, and place the cue ball at various distances away from the object ball. (Use a striped ball in place of the cue ball so that the spin can be easily observed.) For a given distance and shot speed, shoot with just the right amount of backspin so that the cloth friction has time to remove the spin. The cue ball should stop exactly upon impact, and roll afterwards neither forward nor backward. For a fixed distance, the slower the shot speed, the more extreme will be the backspin required to achieve a stun shot impact. Experiment with stun shots on different tables. Sticky tables (high sliding friction between the cloth and ball) require more extreme backspin than slick tables to achieve stun. Stun shots are important for position play and, as discussed in later sections, for judging accurate carom angles.

Problem 2.3: What is the shape of the path taken by a sliding ball before natural roll is achieved? What is the shape of the path after natural roll is achieved?
Answer: Integration of $\mathbf{F} = M\dot{\mathbf{V}}$ twice gives

$$\tfrac{1}{2}\mathbf{F}t^2 = M(\mathbf{q} - \mathbf{q}_0 - \mathbf{V}_0 t)$$

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{V}_0 t + \frac{1}{2M}\mathbf{F}t^2$$

Since the choice of coordinate axes is arbitrary, assume that the axes origin corresponds to t=0, and that the axes are oriented so that the x-component of the sliding force is zero. The coordinates of the path are then given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} V_{0x} \\ V_{0y} \end{pmatrix} t + \frac{1}{2M} t^2 \begin{pmatrix} 0 \\ F_{0y} \end{pmatrix} = \begin{pmatrix} V_{0x} \\ V_{0y} \end{pmatrix} t + \begin{pmatrix} 0 \\ \frac{1}{2}\mu g \end{pmatrix} t^2$$

Because of the choice of axes, the velocity in the x-direction remains unchanged over time. Using the relation $t=x/V_{0x}$ to eliminate t from the y part of this equation gives

$$y = \left(\frac{V_{0y}}{V_{0x}}\right) x + \left(\frac{\mu g}{2V_{0x}^2}\right) x^2$$

which may be recognized as an equation for a parabola. While the ball is sliding on the cloth, the path of the ball is a parabola, the shape of which is determined by the initial velocity and by the frictional force between the ball and the cloth. This path does not depend on the ball mass. This frictional force remains unchanged in both direction and magnitude as long as the ball is sliding. This applies to the paths taken by balls after collisions with cushions or with other balls, and also to the cue ball when struck with an elevated cue stick (*i.e.* masse or semi-masse shots). The ball is accelerated by the sliding force until natural roll is achieved. After natural roll is achieved, there is no sideways force exerted to further accelerate the ball, so the ball rolls in a straight line.

Problem 2.4: When a ball achieves natural roll, what fraction of its kinetic energy is translational and what fraction is rotational?
Answer: The total kinetic energy is

$T = T_{(Trans)} + T_{(Rot)} = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}MV^2 + \tfrac{1}{5}MV^2 = \tfrac{7}{10}MV^2$.

This gives

$$\frac{T_{(Trans)}}{T} = \frac{5}{7}$$

$$\frac{T_{(Rot)}}{T} = \frac{2}{7}$$

Now that the total kinetic energy expression for a natural roll ball is known, the issue of rolling resistance can be examined in more detail. The previous conservative model of a ball rolling up an inclined plane will be used to understand the various forces involved. In the case of a ball rolling without slipping up an inclined plane, the result of these forces is known, namely that $R\omega = V$ is maintained as the ball slows down, but the forces themselves required to achieve this result are not obvious. In order to apply Newton's laws directly, these forces must be known beforehand. Therefore Lagrange's equations of motion will be used. The generalized coordinates will be taken to be the distance up the incline s, the angular rotation of the ball θ, and the undetermined multiplier associated with the constraint equation, λ. The expressions for the kinetic energy, potential energy, and the constraint equation are

$$T = \tfrac{1}{2}MV_s^2 + \tfrac{1}{2}I\omega^2$$

$$U = sMg\sin(\alpha)$$

$$f(s,\theta) = R\theta - s = 0$$

The Lagrangian is $L=T-U+\lambda f$, and the equations of motion are determined from the equation, $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$, for the three coordinates s, θ, and λ. Substitution gives the three equations

$$-Mg\sin(\alpha) - \lambda - M\dot{V}_s = 0$$

$$\lambda R - I\dot{\omega} = 0$$

$$R\theta - s = 0\ .$$

Differentiating the last equation twice gives $R\dot{\omega} = \dot{V}_s$. Solving the second equation for the undetermined multiplier gives $\lambda = I\dot{V}_s/R^2$. Substitution into the first equation then gives

$$M\dot{V}_s = -\left(1 + \frac{I}{MR^2}\right)^{-1} Mg\sin(\alpha) = -\tfrac{5}{7}\, Mg\sin(\alpha)$$

$$= -Mg\sin(\alpha) + \tfrac{2}{7}Mg\sin(\alpha) \qquad \text{[\textit{rolling without slipping}]}$$

$$= F_{gravity} + F_{constraint}$$

If, instead of rolling without slipping, the ball were allowed to slide freely, then Newton's equation of motion in this coordinate system would have been simply

$$M\dot{V}_s = F_{gravity} = -Mg\sin(\alpha) \qquad \text{[\textit{with free slipping}]}$$

Therefore the sliding ball is seen to slow down faster than the rolling ball, all other things being the same. The effective force arising from the static coefficient of friction between the ball and the incline is seen to be $^2\!/_7 Mg\sin(\alpha)$, and this force is directed uphill, opposing the gravitational force. Because there is no sliding associated with this frictional force, there is no energy dissipation in this model system. The only kinetic energy lost is that associated with the corresponding increase in potential energy. As was done in the previous section for a sliding block, an association with the effective slope and a coefficient of friction is made, $\mu_{(rolling)} = \sin(\alpha)$. In the previous section, an equation of motion was assumed of the form $\mu^{eff}_{(rolling)} g = -\dot{V}_s$. It is now seen that this assumption was correct, with the association

$$\mu^{eff}_{(rolling)} = \left(1+\frac{I}{MR^2}\right)^{-1}\sin(\alpha) = \left(1+\frac{I}{MR^2}\right)^{-1}\mu_{(rolling)} = {}^5\!/_7\mu_{(rolling)}$$

When should $\mu^{eff}_{(rolling)}$ be used, and when should $\mu_{(rolling)}$ be used? The answer is that for a rolling billiard ball, it doesn't matter which coefficient of friction is used, provided of course, that it is used with the corresponding equation of motion. The use of the equation of motion involving $\mu_{(rolling)}$ has the advantage that once it has been determined for one object, the same value can be used for other objects made of the same material but with different shapes, such as rolling cylinders, rolling tubes, rings, or hollow balls. The quantity $\mu_{(rolling)}$ is therefore, in some sense, more fundamental than is $\mu^{eff}_{(rolling)}$. The motion of these objects will of course be slightly different, due to the dependence on the moment of inertia of the equations of motion, as demonstrated in the following problem.

Problem 2.5: The table used in P1.5 is moved to the surface of the moon. The billiard ball is replaced with a cylinder made of the same material as a billiard ball. How long will it take for the cylinder to roll the length of the table?

Answer: First determine $\mu_{(rolling)}$ for the table from the previous data:

$$\mu_{(rolling)} = {}^7\!/_5\mu^{eff}_{(rolling)} = \frac{7}{5\;\;96.7} = 0.0145$$

For a solid cylinder, $I=MR^2/2$. $g_{moon}=63.8 in/s^2$, about 1/6 the gravity of the earth. The equation of motion is

$$\dot{V} = -\left(1+\frac{I}{MR^2}\right)^{-1} g_{moon}\mu_{(rolling)}$$

Integration twice over time, then solving for t gives

$$t = \sqrt{\frac{2\left(1+\frac{I}{MR^2}\right)d}{g_{moon}\mu_{rolling}}} = \sqrt{\frac{3\;\;97.75in}{63.8(in/s^2)\;\;0.0145}} = 17.8s$$

Solving the same equation for a ball gives t=17.2s, a result that may also be obtained simply by scaling the earth time, 7.00s by the factor $\sqrt{g_{earth}/g_{moon}} = 2.46$. Therefore, most of the lag time difference is due to the different gravitational forces of the earth and moon, with a smaller difference due to the different moments of inertia of the cylinder and ball.

Problem 2.6: Taking into account both the sliding friction and the rolling resistance, what is the total distance traveled by a cue ball with $V_0=0$ as a function of the initial spin $R\omega_0$? (neglect collisions with other balls and cushions)

Answer: As discussed in more detail in the following sections, $V_0=0$ is the appropriate initial condition immediately after the cue ball collides head-on with an object ball; the object ball removes the velocity of the cue ball, but leaves its spin unchanged. According to P2.2, this spin then accelerates the cue ball to the natural roll velocity $V_{NR}=\frac{2}{7}R\omega_0$. The time required to achieve natural roll is given by

$$t_{NR} = \frac{2R\omega_0}{7\mu_s g}$$

where μ_s is the sliding friction coefficient and ω_0 is taken to be positive. The distance covered by the sliding cue ball during this time is

$$d_{NR} = \tfrac{1}{2}\mu_s g t_{NR}^2 = \frac{2(R\omega_0)^2}{49\mu_s g}.$$

Upon achieving natural roll, the equation of motion is then determined by the rolling resistance. The total rolling time is given by

$$t_R = \frac{V_{NR}}{\mu_{(rolling)}^{eff} g}$$

and the total rolling distance is given by

$$d_R = V_{NR}t_R - \tfrac{1}{2}\mu_{(rolling)}^{eff} g t_R^2 = \frac{V_{NR}^2}{2\mu_{(rolling)}^{eff} g} = \frac{2(R\omega_0)^2}{49\mu_{(rolling)}^{eff} g}.$$

The total distance for both the slide and the roll is

$$d_{total} = d_{NR} + d_R = (R\omega_0)^2 \left(\frac{2}{49g}\right)\left(\frac{1}{\mu_s} + \frac{1}{\mu_{(rolling)}^{eff}}\right).$$

This equation holds for both topspin and draw shots. An important point to notice is that the total distance is proportional to the square of the initial spin. This explains why it is much easier to position the cue ball accurately on a stop shot than on a strong draw or force-follow shot; a small variation in the initial spin is magnified into a larger distance for large initial $R\omega_0$ than for a small initial $R\omega_0$.

3. Cue Tip/Cue Ball Impact

Consider the situation in which a level cue stick strikes the cue ball. The cue tip applies a force to the cue ball at some point on the surface of the ball. This contact time is not instantaneous, but it is very short. Unlike a ball-to-ball impact (characterized by small tangential frictional forces and therefore resulting in a force that is directed essentially between the centers of the balls), the cue tip does not slip on the cue ball (except of course in a miscue situation). With these assumptions, the force is directed along the direction of the cue shaft. The angular acceleration from this force is given by the equation $\mathbf{r} \times \mathbf{F} = \mathbf{I}\,\dot{\omega}$. When a level cue stick strikes the cue ball, the angular acceleration along the direction of force, **F**/|**F**|, is given by

$$\dot{\omega} \cdot \frac{\mathbf{F}}{|\mathbf{F}|} = \left(\mathbf{I}^{-1}(\mathbf{r} \times \mathbf{F})\right) \cdot \frac{\mathbf{F}}{|\mathbf{F}|} = 0 \quad .$$

There is no component of angular acceleration around the axis of the cue stick, so there is no sideways frictional force between the ball and the cloth; the cue ball slides in a straight line in the direction of the cue shaft, while rotating about either or both the vertical axis (*i.e.* sidespin) and the horizontal axis perpendicular to the cue shaft (*i.e.* topspin or draw). This results from the fact that the moment of inertia for a pool ball is proportional to the unit matrix. (If the inertia tensor of an object is not proportional to the unit matrix, e.g. if the ball has an embedded off-center weight, then it will in general curve as it slides or rolls instead of moving in a straight line.)

First consider the case in which the cue tip strikes the cue ball exactly in the center. In this situation $\mathbf{r} \times \mathbf{F} = 0 = \mathbf{I}\,\dot{\omega}$, and there is no angular velocity imparted directly to the cue ball. The only thing that occurs is a transfer of linear momentum and translational energy between the cue stick and the cue ball. It will be assumed that the contact time is so short that the hand/skin/cuestick effects can be ignored. That is, at the very beginning of the contact time, the cue stick slows down and starts moving slower than the hand, and the skin begins to tighten, but by the time any significant extra force is exerted on the cue stick, the cue ball has already departed and lost contact with the cue tip.

Problem 3.1: What is the relation between the cue stick energy and velocity, the length of the stroke, and the applied force? (Assume a constant force is applied by the hand to the cue stick during the stroke.)

Answer: Integration of the equation $F = M_S\dot{V}$ over time gives $Ft = M_S(V - V_0) = M_S V$ where F is the force applied to the stick and M_S is the mass of the cue stick. Integration again gives $\frac{1}{2}Ft^2 = M_S(x - x_0) = M_S d$ in which d is the distance of the stroke. Solving the first equation for t and substitution into the second gives for the kinetic energy

$$T = ½M_S V^2 = Fd.$$

Solving for V gives

$$V = \sqrt{\frac{2Fd}{M_s}}$$

The cue stick energy is proportional to the stroke length and to the applied force, and the cue stick velocity is proportional to the square root of the stroke length and of the applied force. It is important to note that in the expression $T=Fd$, the energy does not depend on the mass of the cue stick. This means that for a given force on the cue stick and a given stroke length, a light cue stick will acquire the same energy as a heavy cue stick.

Problem 3.2: What is the relation between the final cue ball velocity and initial and final cue stick velocity, and the mass of the cue stick?

Answer: Before the impact, only the cue stick has momentum $M_s V_0$ and energy $\frac{1}{2} M_s V_0^2$. After the collision, both the cue stick and the cue ball have energy and momentum. Conservation of momentum and energy, assuming a center-ball impact, give

$$M_s V_0 = M_s V_s + M_b V_b$$

$$\tfrac{1}{2} M_s V_0^2 = \tfrac{1}{2} M_s V_s^2 + \tfrac{1}{2} M_b V_b^2 \quad .$$

Solve the first equation for V_s, and substitute into the second equation to obtain

$$V_b = \frac{2M_s}{M_s + M_b} V_0$$

$$V_s = \frac{M_s - M_b}{M_s + M_b} V_0$$

$$\frac{V_b}{V_s} = \frac{2M_s}{M_s - M_b}$$

A typical cue stick weighs 18*oz*, or about three times the weight of a pool ball. In this case, $V_b = \frac{3}{2} V_0$, $V_s = \frac{1}{2} V_0$, and $V_b/V_s = 3$, so the cue ball is moving about 3 times faster than the cue stick immediately after impact. If the masses were exactly equal (a very light cue stick), then the final ball velocity would be equal to the initial stick velocity, and the final stick velocity would be zero; all of the energy would be transferred from the stick to the ball. If the stick mass were less than the ball mass, then the final stick velocity would be in the opposite direction to the initial stick velocity; that is, the stick would bounce back from the cue ball. Under no condition does $V_b = V_s$; that is, there does not exist a combination of cue stick mass and ball mass such that both are moving forward immediately after impact at the same velocity.

Problem 3.3: What is the fraction of energy that is transferred from the cue stick to the cue ball as a function of the stick and ball masses?

Answer: Using the final stick and ball velocities from P3.2 gives

$$T_b = \tfrac{1}{2} M_b V_b^2 = \frac{4 M_b M_s}{\left(M_s + M_b\right)^2} \left(\tfrac{1}{2} M_s V_0^2\right) = \frac{4 M_b M_s}{\left(M_s + M_b\right)^2} T_0$$

Let $\alpha_s = M_s / M_b$ be the stick to ball mass ratio. Then the ratio of energies is given by

$$\frac{T_b}{T_0} = \frac{4\alpha_s}{(1+\alpha_s)^2}$$

When α_s=1, then this energy ratio is unity, in agreement with the conclusions in P3.2. When there is a mismatch of masses, this energy ratio is less than one and the efficiency of transfer of energy in the collision is reduced.

If a 6*oz* cue stick results in optimal transfer of energy, then why not use one? If it is not optimal, then what is? There are two separate components to the answer. First, it is not always the most efficient transfer of energy that is important, but rather control of the energy that is transferred to the cue ball. It is easier to control a heavier stick than an extremely light one, and the inherent inefficiency from the mass difference is a way to reduce errors in the speed of the cue ball. A possible exception to this is the break shot in open-break games such as 8-ball and 9-ball in which the maximization of cue ball energy is desired. This leads to the second component of the answer.

As the bicep contracts to accelerate the cue stick on the break stroke, both the mass of the forearm and cue stick mass are accelerated. To understand how this affects the final object ball energy in at least a qualitative manner, some simplifying assumptions may be imposed. Assume that the forearm is a thin rod of uniform mass. The moment of inertia of the forearm would be $M_f L^2/3$ where M_f is the mass of the forearm and L is the forearm length. The moment of inertia of the cue stick about the elbow is $M_s L^2$. As both the arm and stick are accelerated about the elbow by a constant force F for an angle θ, the total energy is given by $T=FL\theta$. For a given stroke length $L\theta$ and force F, the total kinetic energy is independent of the cue stick and forearm masses. Writing the two parts of the energy explicitly gives

$$T = T_0 + T_f = \tfrac{1}{2} M_s L\omega^2 + \tfrac{1}{6} M_f L\omega^2 = T_0\left(1 + \frac{M_f}{3M_s}\right)$$

where T_0 is the cue stick energy. Although T, the total kinetic energy of the arm and stick, is fixed by $T=FL\theta$, the fractional division of this energy between the stick and arm is seen to be determined by the mass ratio. It is interesting in this expression that the only important factor is the mass ratio of the forearm and stick; the length of the forearm does not matter, at least within the current set of simplifying assumptions. This means that the optimal cue stick weight will be the same for tall players as for short players, provided the forearm masses are the same. Some players pivot their arm from the shoulder rather than the elbow on the break shot. The above analysis indicates that the additional arm length is irrelevant, but with this technique the entire arm mass rather than simply the forearm mass must be included into the M_f term. Whether this is beneficial or not depends also on the relative forces applied by the different muscle groups involved in the two stroke techniques.

The dilemma is now apparent from the above equation and P3.3. In order to achieve the highest transfer of energy from the cue stick to the cue ball, a very light 6*oz* cue stick would be necessary. But in order to maximize the cue stick energy T_0 for a fixed total energy T during the stroke, a very large cue stick mass would be necessary. Consequently, maximization of the cue ball energy requires some kind of compromise between these two extremes.

The quantity T_0 is the cue stick energy at the end of the stroke, and P3.3 gives the relation between T_0 and the cue ball energy T_b. The combination of these relations gives

$$\frac{T_b}{T} = \frac{4M_b M_s^2}{(M_b + M_s)^2\left(M_s + \frac{1}{3}M_f\right)} = \frac{4\alpha_s^2}{(1+\alpha_s)^2\left(\alpha_s + \frac{1}{3}\alpha_f\right)}$$

In the last expression, $\alpha_s = M_s/M_b$ is the ratio of the stick mass to ball mass, and $\alpha_f = M_f/M_b$ is the forearm to ball mass ratio. For a given forearm mass, the optimum stick mass is determined by differentiating the above expression with respect to α_s, setting the result to zero, and solving for α_s as a function of α_f. The final expression is

$$\alpha_{s(opt)} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{3}\alpha_f}$$

which is an equation for a parabola. When $\alpha_f=0$, it is seen that $\alpha_{s(opt)}=1$, and the optimal cue stick mass would be 6*oz*, a result which agrees with the conclusions from P3.3. A light forearm mass might be 24*oz*, which corresponds to $\alpha_f=4$, $\alpha_{s(opt)}=2.2$, and an optimal cue stick mass of 13.2*oz*. A typical forearm mass might be 36*oz*, which corresponds to an optimal stick weight of 15.4*oz*. A heavy forearm mass might be 64*oz*, which corresponds to an optimal stick weight of 19.3*oz*. A person who breaks with his entire arm, pivoting at the shoulder rather than the elbow, might have an arm mass of 150*oz*, which corresponds to an optimal stick weight of 27.2*oz*.

In the last few years, many professional 9-ball players have switched from heavy break cues to lighter break cues. These players may still use a typical 19-20*oz* cue for their normal strokes in a game, but they break with a lighter 15-18*oz* break cue. Break cues of this weight are consistent with the above equations, elbow pivots rather than shoulder pivots, and slim to medium body types. The actual breaking technique used by these players is more complicated than that considered above, and involves pivots about both the shoulder and the elbow.

Problem 3.4: What is the spin/speed ratio of the cue ball immediately after contact as a function of the vertical cue tip contact point?

Answer: For simplicity assume that the contact point is in the vertical plane through the center of the cue ball. When the cue tip applies a force in an off-center hit, the force accelerates the center of mass, and the resulting momentum is $p=MV$. The linear momentum is given by the expression $p = \int_0^t F(t')dt'$ in which the force is not constant during the contact time and t is the (very short) contact time between the cue tip and the cue ball. (An ideal *impulsive* force is one that integrates to a constant momentum change as the contact time decreases. A cue tip contacting a cue ball and a hammer driving a nail are two examples of nearly ideal impulsive forces.) Integrating the angular acceleration equation in the same way gives $pR\sin(\theta)=pb=I\omega$. The quantity $b=R\sin(\theta)$ is the impact parameter, and is the vertical offset away from a center-ball hit. b is positive for an above-center hit, zero for a center ball hit, and negative for a below-center hit. Eliminating the linear momentum p from these two sets of equations gives

$$MV = \frac{I\omega}{b} = \frac{2MR^2\omega}{5b}$$

$$J = \left(\frac{R\omega}{V}\right) = \frac{5}{2}\left(\frac{b}{R}\right)$$

If b=0, then the angular velocity ω is also zero, which means that there is no spin imparted with a center-ball hit of the cue tip. If the cue tip hits above center, then b is positive and $\omega=\omega_y$ is positive, which means that the ball is rolling in the same direction as the velocity. If the cue tip hits below center, then b is negative and ω is negative, which means that the cue ball is spinning in the opposite direction as in a draw or drag shot. Note that the above equations are valid only for $-R \le b \le R$, or else b is meaningless; the cue tip would miss the cue ball. For practical reasons, b is restricted even more due to the fact that contact points close to the edge of the cue ball result in miscues (see P1.7). Although determined above for angular velocity about the horizontal axis, the same equation applies to angular velocity about the vertical axis resulting from a horizontal impact parameter, or, in fact, to any arbitrary angular velocity axis.

Problem 3.5: At what vertical contact point b_{NR} will the cue ball have natural roll?
Answer: Natural roll occurs when $V=R\omega_y$. Substitution into the above equation gives

$$b_{NR} = \tfrac{2}{5} R$$

Noting that the height above the cloth is given by $z=R+b$, this point may also be written

$$z_{NR} = \tfrac{7}{5} R = \tfrac{7}{10} D$$

where $D=2R$ is the height of the ball. This point is actually rather high on the cue ball, and it is risky to attempt to hit higher than this due to the possibility of miscuing (see P1.7). Sidespin that is imparted to the cue ball with a level stick has no effect on natural roll, so the set of points on the cue ball for which natural roll is achieved immediately with no sliding are along the horizontal line at a height $^7/_{10}$D above the table surface.

Exercise 3.1: Experiment with shots involving natural roll impact points. Use a striped object ball in place of the cue ball. Orient the ball so that the plane defined by the stripe center is tilted at various angles away from vertical. The cue stick should be held as level as possible and should be within the plane defined by the stripe. The cue tip contact point should be exactly in the center of the stripe at a height $^7/_{10}$D above the table. When